Geometry and Physics

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March 6, 2010
Many uses of Mathematics in Physics

• The language of the physical world is mathematics.

• Quantitative understanding of the world around us requires the precise language of mathematics.
• Symmetries play central role in physics. To understand symmetries we need the mathematics of group theory.
• With each continuous symmetry of the system there is an associated conserved quantity (Noether’s Theorem)
• Conservation of momentum follows from translational invariance of the system
• Conservation of angular momentum follows from rotational invariance of the system
And then there are more exotic symmetries leading to new kind of conserved quantities
The Dirac equation, satisfied by the electron, can be obtained from the following Lagrangian density \( (\hbar = c = 1) \):

\[
\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi
\]

- \( \bar{\psi} = \psi^\dagger \gamma^0 \)
- The above Lagrangian is invariant under the following transformation:

\[
\psi \mapsto e^{i\theta} \psi, \quad \bar{\psi} \mapsto e^{-i\theta} \bar{\psi}
\]

- Corresponding to this \( U(1) \) symmetry transformation there is a conserved quantity:

\[
\partial_\mu J^\mu = 0, \quad J^\mu = \bar{\psi} \gamma^\mu \psi
\]

\[
\frac{dQ}{dt} = 0, \quad Q = \int d^3x J^0.
\]

- \( Q \) is the electric charge.
- Conservation of electric charge follows from the invariance of the Lagrangian under a symmetry transformation.
\[ i \hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle \]

- $|\psi\rangle$ is an element of a linear vector space $\mathcal{H}$
- Classical observables (Energy, Momentum, Position, Angular Momentum) become linear Hermitian operators on $\mathcal{H}$
- The eigenvalues of these Hermitian operators become observed values of classical observables
- An eigenvalue $\lambda$ is measured with probability $|\langle \lambda | \psi \rangle|^2$
- $|\lambda\rangle$ is the eigenvector corresponding to the eigenvalue $\lambda$
General Relativity and Riemannian geometry

\[ R_{\mu \nu} - \frac{1}{2}g_{\mu \nu}R = 8\pi G T_{\mu \nu} \]

- Presence of matter changes the geometry of the space
- Requires Riemannian geometry
- Consequences: Black holes, gravitational waves, bending of light...
Quantum Field Theory (Path Integrals) and Functional Analysis

\[ \int \mathcal{D}\phi \, e^{-S[\phi]} \]

- Working with uncountably infinite dimensional function spaces, topology, homotopy groups
- Instantons, Electrically charged objects, Monopoles,...

Example: Standard Model

Based on \( SU(3) \times SU(2)_L \times U(1)_Y \)
Based on $SU(3) \times SU(2)_L \times U(1)_Y$

\[
\begin{align*}
\left(\begin{array}{c}
\nu_e L \\
e_L \\
e_R
\end{array}\right) & \quad (1, 2, -1, 0) \\
\left(\begin{array}{c}
u_e L \\
e_L \\
e_R
\end{array}\right) & \quad (1, 1, -2, -1) \\
\left(\begin{array}{c}
u_e L \\
e_L \\
e_R
\end{array}\right) & \quad (1, 1, -2, -1) \\
\left(\begin{array}{c}
u_e L \\
e_L \\
e_R
\end{array}\right) & \quad (1, 1, -2, -1) \\
\left(\begin{array}{c}
u_e L \\
e_L \\
e_R
\end{array}\right) & \quad (1, 1, -2, -1)
\end{align*}
\]
Complex analysis, conformal mappings, algebraic geometry, group theory,...

Consequences: Ten dimensions, Extended objects, higher dimensional black holes, geometric origin of particles and interactions,...
Many Uses of Physics in Mathematics

Three examples of physics influencing mathematics and leading to new discoveries in mathematics:
1. Supersymmetry
2. Mirror symmetry
3. Knot theory and Statistical Mechanics
Supersymmetry

Consider a massive particle. In the rest frame of the particle

\[ P_\mu = (mc, 0, 0, 0) \]

This vector is invariant under transformations that rotate the last three components (zeroes). These transformation form the group \( SO(3) \subset SO(1, 3) \) called the little group (set of orthogonal matrices with determinant equal to one). Thus massive particles can be labeled by representations of \( SO(3) \). The three generators of rotations \( J_{1,2,3} \) around the three axis satisfy the commutation relation:

\[ [J_a, J_b] = i\epsilon_{abc} J_c \]
A representation of these generators as \((2j + 1) \times (2j + 1)\) matrices exist for all \(j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\).

\(\mathcal{H}_j\): (Spin Space) The vector space on which these generators act for each \(j\) is \((2j + 1)\) dimensional.

A particle which transforms (is rotated) using the representation of the above generators as \((2j + 1) \times (2j + 1)\) matrices is called a spin-\(j\) particle.
• For $j = 0$ we have a spin-0 or a scalar particle (Higgs particle)

\[ J_1 = J_2 = J_3 = 0 \]

The corresponding Hilbert space $\mathcal{H}_0$ is one dimensional $\implies$ scalar particle is described by a single component $\phi(x)$.

• For $j = \frac{1}{2}$ we have spin-$\frac{1}{2}$ particle (electron, muon, tau, quarks)

\[ J_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

These $2 \times 2$ matrices ($\propto$ Pauli matrices) mix the two components needed to describe the particle:

\[ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \]
• For $j = 1$ we have spin-$1$ particle $(W^\pm, Z_0)$

\[
J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

These $3 \times 3$ matrices now mix the three components needed to describe the particle:

\[
\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}
\]

• The particles with spin $j = 0, 1, 2, 3, \cdots$ are called Bosons
• The particles with spin $j = \frac{1}{2}, \frac{3}{2}, \cdots$ are called Fermions
Bosons and Fermions behave very differently. Consider rotating a spin-\(j\) particle by an angle \(\theta\) around an axis say \(z - \text{axis}\). The matrix that does this is

\[
g_j(\theta) = e^{i\theta} J_3 = \begin{pmatrix}
  e^{i\theta} j & 0 & 0 & \cdot & \cdot & 0 \\
  0 & e^{i\theta} (j-1) & 0 & \cdot & \cdot & 0 \\
  0 & 0 & e^{i\theta} (j-2) & \cdot & \cdot & 0 \\
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  0 & 0 & 0 & 0 & e^{i\theta} (j-3) & \cdot \\
  \cdot & \cdot & \cdot & \cdot & \cdot & e^{i\theta} (-j)
\end{pmatrix}
\]

\[
g_j(2\pi) = \begin{cases}
  1 & j = 0, 1, 2, \ldots \\
  -1 & j = \frac{1}{2}, \frac{3}{2}, \ldots
\end{cases}
\]

Bosons come back to themselves after a rotation by \(2\pi\) but fermions do not!

Pauli’s exclusion principle holds for fermions but not for bosons.
Because they behave so differently it seems unlikely that a symmetry can exist which converts bosons into fermions and fermions into bosons.

Also there is a no-go theorem (Coleman-Mandula theorem) which forbids conserved charges which are not Lorentz invariant other than those belonging to the Poincare’ group (such as momentum).

But if a charge is Lorentz invariant it can not change the spin hence can not change a fermion into a boson.

Coleman-Mandula theorem only talked about conserved charges coming from symmetries whose generators satisfied the commutation relations:

\[
[T_a, T_b] = f_{abc} T_c
\]

In the 1970s physicists came up with the idea of supersymmetry, a symmetry in which some generators satisfied commutation relations and some satisfied anti-commutation.
\[
[T_a, T_b] = f_{abc} T_c \\
\{Q_a, Q_b\} = g_{abc} T_c \\
[Q_a, T_b] = k_{abc} Q_c
\]

Mathematicians have extensively studied this structure and have formalized it under the name of supergroups.

The simplest example of a supersymmetric system is the following: Let \(a\) and \(b\) be two operators such that

\[
[a, a^\dagger] = 1 \quad \{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1 \\
[a, a] = [a^\dagger, a^\dagger] = 0 \quad \{b, b\} = \{b^\dagger, b^\dagger\} = 0 \\
[a, b] = 0 \quad [a, b^\dagger] = 0
\]
The operator $a$ and $a^\dagger$ are just harmonic oscillator annihilation and creation operators. The Hamiltonian is given by $H_B = a^\dagger a$ and eigenstates of the Hamiltonian are $\{ |0_B\rangle, |1_B\rangle, |2_B\rangle, \cdots \}$ such that

$$a|0_B\rangle = 0, \quad |n_B\rangle \propto (a^\dagger)^n |0_B\rangle$$

$$H_B |n_B\rangle = n |n_B\rangle$$

The operator $b$ and $b^\dagger$ define a fermionic harmonic oscillator. The Hamiltonian is $H_F = b^\dagger b$ and eigenstates of the Hamiltonian are

$$|0_F\rangle, \quad b^\dagger |0_F\rangle$$

$$b |0_F\rangle = 0$$

For the combined system the total Hamiltonian is $H = H_B + H_F$ and the eigenstates are $\langle 0 | = |0_B\rangle \langle 0_F |$

$$|0\rangle \quad |1\rangle = a^\dagger |0\rangle \quad |2\rangle = (a^\dagger)^2 |0\rangle \quad |3\rangle = (a^\dagger)^3 |0\rangle \quad \cdots$$

$$b^\dagger |0\rangle \quad b^\dagger |1\rangle \quad b^\dagger |2\rangle \quad \cdots$$
\[ \mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_F \]

\( \mathcal{H}_B \) has basis states \( \{ |0\rangle_B , a^\dagger |0\rangle_B , (a^\dagger)^2 |0\rangle_B , (a^\dagger)^3 |0\rangle_B \cdots \} \)

\( \mathcal{H}_F \) has basis states \( \{ |0\rangle_F , b^\dagger |0\rangle_F \} \)

\[
\text{Tr}_{\mathcal{H}} e^{-\beta H} = 1 + 2 e^{-\beta} + 2 e^{-2\beta} + 2 e^{-3\beta} + \cdots \\
= 1 + e^{-\beta} \\
= \frac{1}{1 - e^{-\beta}}
\]

\[
\text{Tr}_{\mathcal{H}_B} e^{-\beta H} = 1 + e^{-\beta} + e^{-2\beta} + e^{-3\beta} + \cdots \\
= \frac{1}{1 - e^{-\beta}}
\]

\[
\text{Tr}_{\mathcal{H}_F} e^{-\beta H} = 1 + e^{-\beta}
\]
The operator $F = b^\dagger b$ is 0 on bosonic states and +1 on fermionic states. It can distinguish the two states.

$$\text{Tr}_\mathcal{H}(-1)^F e^{-\beta H} = 1 + (1 - 1) e^{-\beta} + (1 - 1) e^{-2\beta} + (1 - 1) e^{-3\beta} + \cdots = 1$$

This is called the Witten index. The operator $Q := a^\dagger b + b^\dagger a$ is such that $Q^2 = H$ and

$$[Q, a] = -b \quad [Q, a^\dagger] = b^\dagger \quad \{ Q, b \} = a \quad \{ Q, b^\dagger \} = a^\dagger$$

- In physics supersymmetry cancels divergences and make QFT (with enough supersymmetry) finite.
- In mathematics invariants of different kind are formulated in terms of Witten index (Atiyah-Singer index theorem)
- Supersymmetric QFTs are used to study number theory problems (Langlands program)
- Knot invariants are formulated in terms of Witten index
Mirror Symmetry

Consider a free point particle moving on a cylinder.

The state of the particle in the position basis is

$$ |\psi\rangle = \int dx dy \langle x, y | \psi \rangle | x, y \rangle $$

Probability amplitude of finding the particle in the region $dx dy$ around $(x, y)$

$$ \psi(x, y) := \langle x, y | \psi \rangle $$

is the solution of the Schrödinger equation

$$ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y) = \left( \frac{p_x^2 + p_y^2}{2m} \right) \psi(x, y) $$
The wave function of the free particle is

\[ \langle x, y \mid \psi \rangle \propto \exp \left( i \frac{p_x x}{\hbar} + i \frac{p_y y}{\hbar} \right) \]

\( y \) is periodic with period \( 2\pi R \Rightarrow p_y = \frac{n\hbar}{R} \), \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \)

Momentum in the compact direction is quantized.

\[ E = \frac{p_x^2 + p_y^2}{2m} = \frac{p_x^2}{2m} + n^2 \frac{\hbar^2}{2mR^2} \]

- \( R \mapsto 0 \) is very different from \( R \mapsto \infty \)
Now consider a string (a loop) on a cylinder.

The basis of the Hilbert space of the string is

$$|x, y; w\rangle$$

$$(x, y) \in \mathbb{R} \times S^1, \quad w \in \mathbb{Z}$$

- $(x, y)$ is the position of the center of mass of the string
- $w$ is the winding number
The energy is given by

\[ E = \frac{p_x^2}{2m} + n^2 \frac{\hbar^2}{2m R^2} + \frac{k}{2} \left( w \frac{2\pi R}{L} \right)^2 \]

\text{winding energy}

- \( k = \frac{\hbar^2}{4\pi^2 m \alpha'^2} \)
- \([\alpha'] = L^{-2}\)

\[ E = \frac{p_x^2}{2m} + \frac{\hbar^2}{2m} \left( \frac{n^2}{R^2} + w^2 \alpha'^2 R^2 \right) \]

- \( R \leftrightarrow \frac{1}{\alpha' R} \) and \( n \leftrightarrow w \)
- Strings see geometry very differently than point particles.
- This symmetry is called T-duality.
Strings on a torus

- $T^2 = S^1 \times S^1$
- The torus is determined by two parameters: $R_1$ and $R_2$ (The radius of 1st and 2nd circle respectively)

$$A = i \alpha' R_1 R_2, \quad \tau = i \frac{R_2}{R_1}$$

- $A$ determines the size of the torus ($\propto$ area of the torus)
- $\tau$ determines the shape of the torus
- $R_1 \leftrightarrow R_2 \implies A \leftrightarrow A$
- $R_1 \leftrightarrow R_2 \implies \tau \leftrightarrow -\frac{1}{\tau}$ (This is part of the larger symmetry of the torus: $SL(2, \mathbb{Z})$)
\[ A = i \alpha' R_1 R_2, \quad \tau = i \frac{R_2}{R_1} \]

- \( R_1 \mapsto \frac{1}{\alpha' R_1} \)

\[ A = i \alpha' R_1 R_2 \mapsto i \frac{R_2}{R_1} = \tau, \quad \tau = i \frac{R_2}{R_1} \mapsto i \alpha' R_1 R_2 = A \]

- The size and shape parameters get interchanged
- Simplest example of Mirror Symmetry
In general a space might have many size parameters (Kähler parameters) and many shape parameters (Complex structure parameters)

Strings of one type on $X \cong$ Strings of another kind on $Y$

Kähler parameters of $X \Leftrightarrow$ Complex structure parameters
Complex structure parameters of $X \Leftrightarrow$ Kähler parameters of $Y$

- The study of such a pair of spaces (called mirror manifolds) is a very active field of research for the past many years.
• Two knots are equivalent if one can be deformed into the other without breaking it (Homeomorphism)
• The main problem in knot theory is how to determine if the two given knots are equivalent
• Knots live in three dimensions but they can be projected into a plane
• Reidemeister moves are the generators of the deformations:

• One way to distinguish two knots is to define a function (called knot invariant) on the knots such that the value of the function does not change as we deform the knot.
• Such a function will divide the space of knots into disjoint sets such that the function takes a constant value on a set. This implies that two knots in different sets can not be deformed into each other and hence are not equivalent. This, however, does not mean that two knots in a given set (on which the function takes a constant value) are equivalent. The quest is to find a function which takes different values on inequivalent knots.
• One knot invariant is the minimum number of crossings in a projection of the knot.
Another invariant can be defined as follows:
\[
\sqrt{0} = 1 \\
\sqrt{00} = -(2 + \sqrt{2})^{1/2} \sqrt{2}
\]
\[
\sqrt{V(\mathbb{Z})} - 2 \sqrt{V(\mathbb{R})} = (2 - \sqrt{2}) \sqrt{51}
\]
\[
\sum (-1)^q \# \text{ of unknots}
\]
M. Khovanov

\[ \sum (-1)^k g \]

looks like Euler characteristic of something.

\[ \sum t^g \]

\[ \begin{cases} \text{Physics} \\ \sum (-1)^k g \end{cases} \Rightarrow \text{Tree } (-1)^F e^H \]

\[ \sum t^g = \text{Tree } tF e^H \]

\[ g = \bar{e} \]
In the last twenty years physics (string theory, supersymmetric gauge theories) have provided a wealth of new concepts in mathematics. There is no reason to believe this amazing symbiotic relationship will end soon.

Join in and enjoy the ride.