Recitation 1: Solution

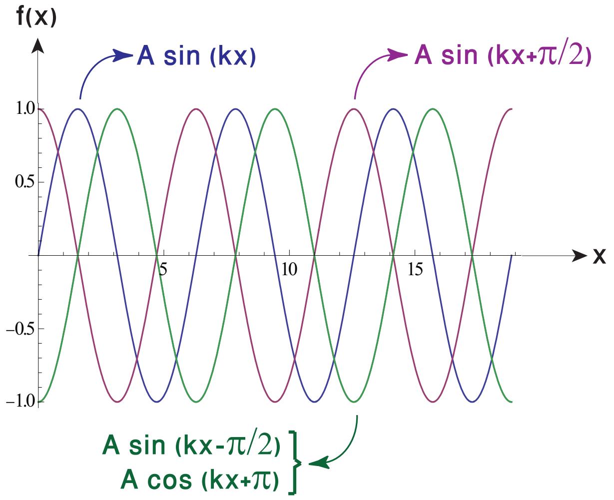
1. (a) Draw the following waves on top of each other, on the same axis.

$$\begin{aligned} f_1(x) &= A\sin(kx) \\ f_2(x) &= A\sin\left(kx + \frac{\pi}{2}\right) \\ f_3(x) &= A\sin\left(kx - \frac{\pi}{2}\right) \\ f_4(x) &= A\cos(kx + \pi). \end{aligned}$$

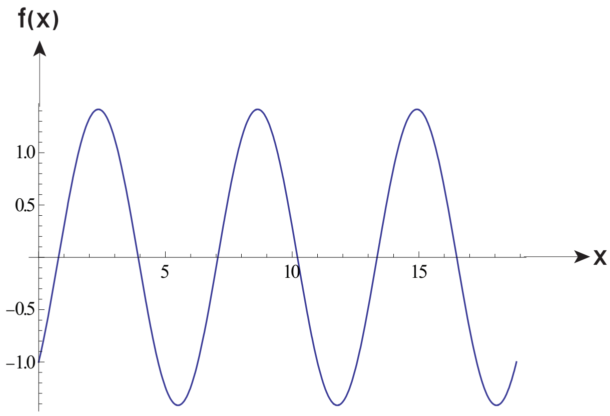
   (b) Draw the resultant wave that is obtained by adding (superposing) f_1 and f_2.

Answer 1: (a)

$$\begin{aligned} f_1(x) &= A\sin(kx) = A\sin[k(x+0)], && \text{starting point is zero.} \\ f_2(x) &= A\sin\left(kx + \frac{\pi}{2}\right) = A\sin\left[k\left(x + \frac{\pi}{2k}\right)\right] \\ &= A\sin\left[k\left(x + \frac{\pi}{2}\cdot\frac{\lambda}{2\pi}\right)\right] = A\sin\left[k\left(x + \frac{\lambda}{4}\right)\right], && \text{starting point is } -\frac{\lambda}{4}. \\ f_3(x) &= A\sin\left(kx - \frac{\pi}{2}\right) = A\sin\left[k\left(x - \frac{\pi}{2k}\right)\right] \\ &= A\sin\left[k\left(x - \frac{\pi}{2}\cdot\frac{\lambda}{2\pi}\right)\right] = A\sin\left[k\left(x - \frac{\lambda}{4}\right)\right], && \text{starting point is } +\frac{\lambda}{4}. \\ f_4(x) &= A\cos(kx+\pi) = A\cos\left[k\left(x + \frac{\pi}{k}\right)\right] \\ &= A\cos\left[k\left(x + \pi\cdot\frac{\lambda}{2\pi}\right)\right] = A\cos\left[k\left(x + \frac{\lambda}{2}\right)\right], && \text{starting point is } -\frac{\lambda}{2}. \end{aligned}$$



(b) Resultant wave will be,



2. Consider two waves,

$$f_1(x,t) = A\sin(k_1 x - \omega_1 t)$$
$$f_2(x,t) = A\sin(k_2 x - \omega_2 t),$$

where $k_2 = k_1 + \Delta$, where Δ is very small. i.e., the wavelengths of the two waves are very close to each other. By superposing the two waves, find a mathematical expression for the resultant wave that would allow you to draw a snapshot of the wave accurately. Draw the snapshot. This is the phenomena of beats.

Answer 2:

We are given that,

$$f_1(x,t) = A\sin(k_1 x - \omega_1 t)$$
$$f_2(x,t) = A\sin(k_2 x - \omega_2 t),$$

where $k_2 = k_1 + \Delta$. Let $f(x,t)$ is the resultant wave that is obtained by the superposition of two given waves, then according to superposition principle,

$$\begin{aligned} f(x,t) &= f_1(x,t) + f_2(x,t) \\ &= A\sin(k_1 x - \omega_1 t) + A\sin(k_2 x - \omega_2 t) \\ &= A[\sin(k_1 x - \omega_1 t) + A\sin(k_2 x - \omega_2 t)]. \end{aligned}$$

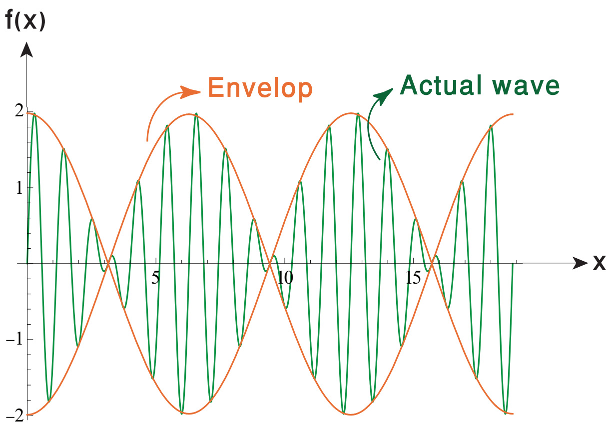
Since according to sum to product formula $\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$,

$$\begin{aligned} f(x,t) &= A\left[2\sin\left(\frac{(k_1x-\omega_1t)+(k_2x-\omega_2t)}{2}\right)\cos\left(\frac{(k_1x-\omega_1t)-(k_2x-\omega_2t)}{2}\right)\right] \\ &= 2A\sin\left(\frac{(k_1+k_2)x-(\omega_1+\omega_2)t}{2}\right)\cos\left(\frac{(k_1-k_2)x-(\omega_1-\omega_2)t}{2}\right) \\ &= 2A\sin\left(\frac{(k_1+k_1+\Delta)x-(\omega_1+\omega_2)t}{2}\right)\cos\left(\frac{(k_1-k_1-\Delta)x-(\omega_1-\omega_2)t}{2}\right) \\ &= 2A\sin\left(\frac{(2k_1+\Delta)x-(\omega_1+\omega_2)t}{2}\right)\cos\left(\frac{-\Delta x-(\omega_1-\omega_2)t}{2}\right). \end{aligned}$$

Let's freeze the wave with respect to time or in other words take a snapshot at $t=0$ of the wave. Resultant wave will be,

$$\begin{aligned} f(x) &= 2A\sin\left(\frac{(2k_1+\Delta)x}{2}\right)\cos\left(\frac{-\Delta x}{2}\right) \\ &= 2A\sin\left[\left(k_1+\frac{\Delta}{2}\right)x\right]\cos\left(\frac{\Delta x}{2}\right). \end{aligned}$$

Snapshot of the resultant wave is given below.



3. Consider the wave equation that we discussed in the class

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2},$$

where this time we consider the quantity that is waving to be complex function of space and time. i.e., $f(x,t)$ gives us a complex number for each position x in space at a given time t.

(a) Show that $f(x,t) = Ae^{i(kx-\omega t)}$ is a solution. What is the condition on ω and k for this to be a solution?

(b) What is the "wavelength" and "time period" of this solution? What is the speed?

Answer 3:

We are given that,

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}.$$

We want to prove that $f(x,t) = Ae^{i(kx-\omega t)}$ is a solution to the given equation.

$$\begin{aligned} f(x,t) &= Ae^{i(kx-\omega t)} \\ \frac{\partial f}{\partial t} &= Ae^{i(kx-\omega t)}(-i\omega) \\ \frac{\partial^2 f}{\partial t^2} &= Ae^{i(kx-\omega t)}(-i\omega)^2 \\ \frac{\partial^2 f}{\partial t^2} &= -A\omega^2 e^{i(kx-\omega t)}. \end{aligned} \tag{1}$$

Similarly

$$\begin{aligned} \frac{\partial f}{\partial x} &= Ae^{i(kx-\omega t)}(ik) \\ \frac{\partial^2 f}{\partial x^2} &= Ae^{i(kx-\omega t)}(ik)^2 \\ \frac{\partial^2 f}{\partial x^2} &= -Ak^2 e^{i(kx-\omega t)}. \end{aligned}$$

Multiply both sides of above equation by c^2 we get,

$$\begin{aligned} c^2 \frac{\partial^2 f}{\partial x^2} &= -Ak^2 c^2 e^{i(kx-\omega t)} \\ &= -A(kc)^2 e^{i(kx-\omega t)}. \end{aligned} \tag{2}$$

Comparison of equation (1) and equation (2) gives the condition for $f(x,t) = Ae^{i(kx-\omega t)}$ to be a solution to the given equation. $\boxed{\omega = ck}$ is the condition for this to be a solution.

(b) The wavelength of this solution will be $\lambda = 2\pi/k$. The time period will be $T = 2\pi/\omega$ and the speed is $c = \omega/k$.

4. Consider another wave equation

$$i\frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t),$$

$\hbar$ and m, for us are just some constants for now.

Show that $\psi(x,t) = Ae^{i(kx-\omega t)}$ is a solution of this equation, what is the relation between k and ω for this to be a solution?

Answer 4:

We are given that,

$$i\frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)\cdot$$

We want to prove that, $\psi(x,t) = Ae^{i(kx-\omega t)}$ is a solution to this equation.

$$\begin{aligned}
\psi(x,t) &= Ae^{i(kx-\omega t)} \\
\frac{\partial \psi}{\partial t} &= Ae^{i(kx-\omega t)}(-i\omega) \\
\frac{\partial \psi}{\partial t} &= -iA\omega e^{i(kx-\omega t)} \\
i\frac{\partial \psi}{\partial t} &= A\omega e^{i(kx-\omega t)}. \qquad (3)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= Ae^{i(kx-\omega t)}(ik) \\
\frac{\partial^2 \psi}{\partial x^2} &= Ae^{i(kx-\omega t)}(ik)^2 \\
\frac{\partial^2 \psi}{\partial x^2} &= -Ak^2 e^{i(kx-\omega t)} \\
-\frac{\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} &= -\frac{\hbar}{2m}\cdot -Ak^2 e^{i(kx-\omega t)} \\
-\frac{\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} &= A\frac{\hbar k^2}{2m}e^{i(kx-\omega t)}. \qquad (4)
\end{aligned}$$

Comparison of equation (3) and equation (4) gives the condition for $\psi(x,t) = Ae^{i(kx-\omega t)}$ to be a solution to given equation. Thus $\psi(x,t) = Ae^{i(kx-\omega t)}$ is a solution to given equation iff $\boxed{\omega = \frac{\hbar k^2}{2m}}$.