Recitation 3
Fourier Having Fun With Waves

1. Consider the functions of the following form

$$\cos(\frac{n\pi}{L}x), \quad \sin(\frac{n\pi}{L}x) \quad n = 0, 1, 2, 3, \cdots$$

where L is any constant. Once you chose a particular value of L you have a set of these different waves with an infinite members in the set. Verify the following curious properties of these functions (waves):

(a)

$$\frac{1}{L}\int_{-L}^{L} \cos(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)\ dx = 0 \quad \text{if } m \neq n$$
$$= 1 \quad \text{if } m = n$$

(b)

$$\frac{1}{L}\int_{-L}^{L} \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x)\ dx = 0 \quad \text{if } m \neq n$$
$$= 1 \quad \text{if } m = n$$

(c)

$$\frac{1}{L}\int_{-L}^{L} \sin(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)\ dx = 0 \quad \text{for any } m, n$$

2. What is the smallest wavenumber present in these waves of the above set? What is the corresponding largest wavelength? What is the change in wavenumber in successive members as we change n by one?

3. Suppose we make a general superposition using waves from this set i.e.,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x) + \sum_{n=0}^{\infty} B_n \cos(\frac{n\pi}{L}x),$$

where A_n and B_n are telling us the strengths of corresponding waves in the superposition. Show that any function of this form is periodic with a period of $2L$.

4. Now suppose that somebody makes a superposition of their desire using some of these waves and tells you only the resultant function $f(x)$. Show that you can find the secret individual strengths for each wave from the set by using the formulas:

$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x), \quad B_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)$$

5. Fourier had the bright idea that you can pretty much get any function that you desire, which is periodic with period $2L$, using some superposition of the above set of waves. Find the superposition which gives you a saw tooth wave, i.e.,

$$\begin{aligned} f(x) =& x, \quad -L < x < L \\ & \text{then repeates itself with a period of } 2L \end{aligned} \tag{1}$$