Problem Set 1: Waves, Light and Interference

Solution

1. A harmonic plane wave in a water pond is moving in a particular direction that we can call x-axis. The wave has an amplitude of 0.5 m and a wavelength of 0.25 m whereas its speed is 2 m/s. What is the angular frequency and angular wave number of this wave? How many waves fit in a length of 10 m. Suppose at $t = 0$ there is a full peak at $x = 0$. What is the height of water at $x = 2$ m and $x = 2.1$ m at this time? What is the height at these same points at $t = 1$ s?

   Answer 1:

   Recall that a harmonic plane wave with amplitude A, angular wave number k, angular frequency ω and phase ϕ moving in the positive x- direction is given by the following function of position and time

$$f(x,t) = A\sin(kx - \omega t + \phi).$$

   We are given that this wave has amplitude $A = 0.5$ m, wavelength $\lambda = 0.25$ and speed $v = 2$ m/s. From these values we can find ω and k by formulas derived in class.

$$\begin{aligned} k &= \frac{2\pi}{\lambda} = \frac{2\pi}{0.25 \text{ m}} = 8\pi \text{ m}^{-1}, \\ \omega &= vk = (2 \text{ m/s})(8\pi \text{ m}^{-1}) = 16\pi \text{ s}^{-1}. \end{aligned}$$

   So that wave gets the form

$$f(x,t) = 0.5 \ \sin(8\pi x - 16\pi t + \phi) \tag{1}$$

   To find the value of ϕ, we use the initial condition. Wave has the maximum value when $x = t = 0$. Maximum value for $f(x,t)$ in equation (**??**) is 0.5, so by plugging in values for this condition we get

$$\begin{aligned} 0.5 &= 0.5\sin(8\pi(0) - 16\pi(0) + \phi) \\ \Rightarrow \sin\phi &= 1 \\ \Rightarrow \quad \phi &= \frac{\pi}{2}. \end{aligned}$$

Therefore $f(x,t)$ has following form

$$f(x,t) = 0.5\sin\left(8\pi - 16\pi t + \frac{\pi}{2}\right)$$

Now we can find height of this wave at different values of x and t.

At $t = 0$, $x = 2$ m:

$$\begin{aligned} f(2,0) &= 0.5\sin(16\pi(0) + \pi/2) \\ &= 0.5 \end{aligned}$$

At $t = 0$, $x = 2.1$ m:

$$\begin{aligned} f(2.1,0) &= 0.5\sin(16\pi(2.1) + \pi/2) \\ &= -0.4045 \end{aligned}$$

At $t = 1$ s, $x = 2$ m:

$$\begin{aligned} f(2,1) &= 0.5\sin(8\pi(2) - 16\pi(1) + \pi/2) \\ &= 0.5 \end{aligned}$$

At $t = 1$ s, $x = 2.1$ m:

$$\begin{aligned} f(2.1,1) &= 0.5\sin(8\pi(2.1) - 16\pi(1) + \pi/2) \\ &= -0.4045 \end{aligned}$$

2. Consider two waves that are superposing

$$\begin{aligned} f_1(x) &= A\sin(kx) \\ f_2(x) &= A\sin\left(kx + \frac{\pi}{4}\right). \end{aligned}$$

What is the wavelength of the resultant wave? What is the amplitude of the resultant wave?

Answer 2:

The resultant wave $f_{\text{res}}(x)$ can be found by adding the two waves given in the question: $f_1(x)$ and $f_2(x)$.

$$\begin{aligned} f_{\text{res}}(x) &= f_1(x) + f_2(x) \\ &= A\left[\sin(kx) + \sin\left(kx + \frac{\pi}{4}\right)\right] \end{aligned}$$

Now I can use the trigonometric identity,

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

to get the following form for $f_{\rm res}(x)$.

$$\begin{aligned} f_{\rm res}(x) &= 2A\sin\left(\frac{kx + kx + \pi/4}{2}\right)\cos\left(\frac{kx - kx - \pi/4}{2}\right) \\ &= 2A\sin\left(kx + \frac{\pi}{8}\right)\cos\left(-\frac{\pi}{8}\right) \\ &= 2A\cos\left(\frac{\pi}{8}\right)\sin\left(kx + \frac{\pi}{8}\right) \\ \Rightarrow f_{\rm res}(x) &= 1.848\ A\sin\left(kx + \frac{\pi}{8}\right). \end{aligned} \tag{2}$$

where in the last step I have plugged in value of $\cos(\pi/8) = 0.924$.

Now, from equation (**??**) we see that $f_{\rm res}(x)$ is a wave with amplitude 1.848 A and phase $\pi/8$. The wavenumber of this wave is being represented by k and so its wavelength is $\lambda = 2\pi/k$.

3. **Speakers in a gathering**

Two speakers are placed at a distance of 20 meters from each other. Consider the line joining the two speakers. Let us set up a coordinate system so that this line is the x-axis and speaker 1 is placed at $x = 0$. Then, the other speaker is placed at $x = 20$ m. Both speakers are given same electric signal and they are producing static sound consisting of harmonic wave of frequency 1000 Hz.

(a) Write down an expression for the sound wave from speaker 1 along the x-axis such that at $t = 0$ the wave starts as a pure sine wave from $x = 0$ with out any phase shift. Speed of sound is 340 m/s.

(b) Now right down an expression for the wave from second speaker. Notice that the same feed is being given to the second speaker. Hence you can get the expression for second speaker by translating the first wave to the right by 20 m. How would you do that. Also, this wave is moving in opposite direction, but you can take care of that by reversing the relative sign of x and t terms in the sine. Also, be careful that the wave starts in the ascending fashion to the left. How will you make that sure?

(c) Now let us look at this wave at a fixed time, say, $t = 0$. On what spots on the line will you hear the loudest sound? On what spots will you hear the faintest sound?

Answer 3:

(a) We want to write expression for sound wave with frequency $f = 1000$ Hz and speed $v = 340$ m/s. The angular frequency ω is related to frequency f by $\omega = 2\pi f$. So

$$\omega = 2\pi f = 2000\pi \text{ rad/s.}$$

and we can find wavenumber k by using $\omega = vk$.

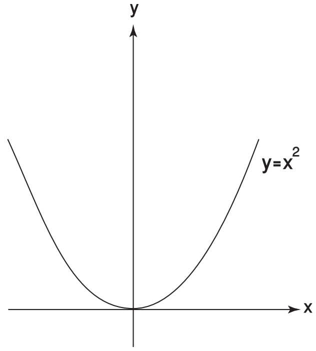
$$k = \frac{\omega}{v} = \frac{2000\pi}{340} = 18.5$$

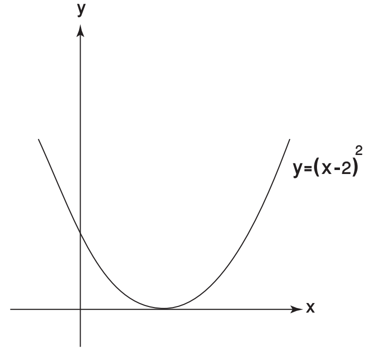
Also, since there is no phase shift the sound wave emitted by speaker 1 has following exact form

$$\begin{aligned} f_1(x,t) &= A\sin(kx - \omega t) \\ f_1(x,t) &= A\sin(18.5x - 2000\pi t) \end{aligned} \quad (3)$$

(b) For the wave emitted by the second speaker, the speed of sound and frequency are the same which implies that ω and k will remain unchanged. We, however, have to make adjustments because of the following three reasons.

i. The second speaker is present at $x = 20$ m instead of $x = 0$. So we will have to shift our sine wave by 20 units on x-axis. Remember that to shift a function $f(x)$ "a" units to the right, we let $x \to x - a$ in the argument of function. So for example, to shift graph of $f(x) = x^2$, 2 units to the right, I let $x \to x - 2$ and draw $f(x) = (x - 2)^2$.





Similarly to translate the wave to the right by 20 m, I let $x \to x - 20$ in equation (**??**) and call the new function $f_2(x,t)$.

$$f_2(x,t) = A\sin(18.5(x-20) - 2000\pi t) \tag{4}$$

ii. We need to take care of the fact that the sound wave from speaker 2 is moving to the left so we should switch the relative sign between x and t in equation (**??**).

$$f_2(x,t) = A\sin(18.5(x-20) + 2000\pi t) \tag{5}$$

iii. If we plot the graph of $f_2(x,t)$ at $t = 0$, we find that right to the left of $x = 20$, the wave $f_2(x,0)$ attains negative values. We want the wave to be ascending in the left direction as the speakers are turned on. To turn the negative values into positive ones we should multiply $f_2(x,t)$ in equation (**??**) with a minus sign. So the final form of sound wave emitted by speaker 2 is

$$f_2(x,t) = -A\sin(18.5(x-20) + 2000\pi t)$$

(c) To find the resultant wave, I add the sound waves emitted by both speakers.

$$\begin{aligned} f_{\text{res}}(x,t) &= f_1(x,t) + f_2(x,t) \\ &= A[\sin(18.5x - 2000\pi t) - \sin(18.5(x-20) + 2000\pi t)] \\ &= A\sin\left[\frac{20 \times 18.5}{2} - \frac{(2000+2000)\pi t}{2}\right]\cos\big[18.5x - 10 \times 18.5\big] \\ &= -A\cos\big[18.5x - 185\big]\sin\big[2000\pi t + 185\big]. \end{aligned}$$

The sine factor tells us that standing at any point on x-axis, how the sound wave varies with time and essentially has information about frequency of sound. However, the cosine factor tells us what would be the amplitude of this oscillation at various points on x-axis.It is fixed as a function of position as it has no dependance on time. Hence where we have maximum of this function, we will have louder sound because of larger oscillations. Similarly, we will hear faint sound when there are smaller oscillations. Therefore for loud sound,

$$\cos\big[18.5x - 185\big] = \pm 1.$$

The sound will be loudest at the crests and troughs of $f_{\text{res}}(x,0)$, i.e. when argument of cosine is $n\pi$, for n an integer, i.e.,

$$\begin{aligned} 18.5x_{\text{loud}} - 185 &= n\pi \\ \Rightarrow x_{\text{loud}} &= \frac{n\pi + 185}{18.5}, \end{aligned}$$

and the sound will be faintest when the height of $f_{\text{res}}(x,0)$ is zero, i.e. when the argument of cosine is $\left(n+\frac{1}{2}\right)\pi$; for n an integer.

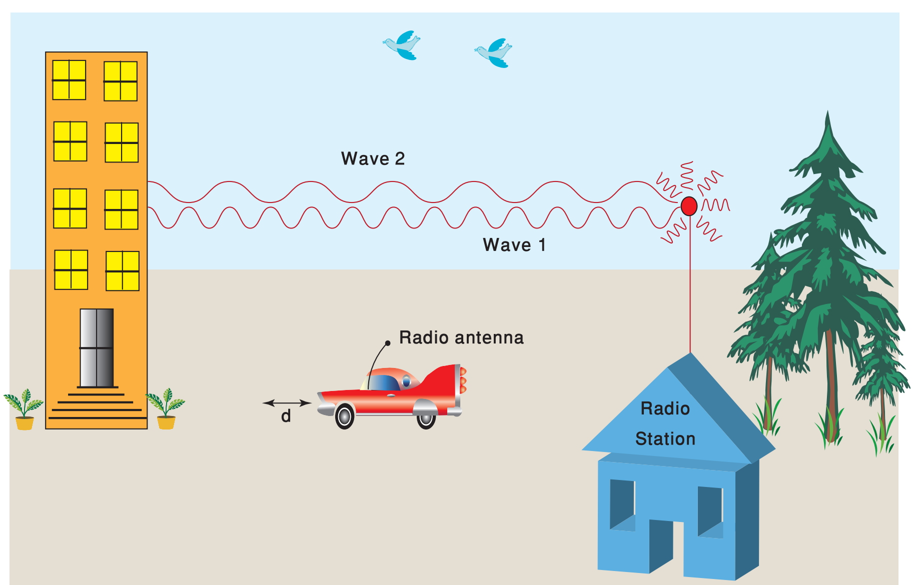
$$\begin{aligned} 18.5x_{\text{faint}} - 185 &= \left(n+\frac{1}{2}\right)\pi \\ \Rightarrow x_{\text{faint}} &= \frac{\left(n+\frac{1}{2}\right)\pi + 185}{18.5}, \end{aligned}$$

4. **FM Radios**

   (a) While listening to your favorite FM channel in a parked car, you must have noticed that sometimes the reception is affected if you displace your car slightly. Keeping in mind the phenomena of interference and possibility of reflection of radio waves from nearby objects, explain why this could be happening?

   (b) For a channel broadcasted at a frequency of 100 Mhz, by what minimum distance you must move your car (in an appropriate direction) for the signal to become pronounced, if you start from a spot of diminished signal?

Answer 4:



(a) An FM radio works by receiving electromagnetic waves from a nearby radio station. In the presence of objects near your parked car, the EM waved would be reflected off those objects and then interfere together at the antenna of radio. By displacing your car slightly, you let the optical path length of the interfering waves vary which would cause a change in the way these waves interfere at the antenna. Thus a change in radio transmission could be heard depending on if we got a maximum or minimum.

(b) The frequency of the electromagnetic wave is $f = 100$ MHz$= 10^8$ Hz. We want to find the minimum distance that the car moves such that the destructive interference of two waves changes completely into their constructive interference. Let the distance that the car travels be d. Then each wave covers a distance that varies from its initial path length by d. We have following expression for changing destructive interference into constructive interference.

$$\begin{aligned} 2d &= \frac{\lambda}{2} \\ \Rightarrow d &= \frac{\lambda}{4}, \end{aligned} \tag{6}$$

where λ is the wavelength of the EM wave for which frequency is 10^8 Hz. For EM wave we have $\omega = ck$, where $c = 3 \times 10^8$ m/s is the speed of light. Using $k = 2\pi/\lambda$ and $\omega = 2\pi f$ we can find following relation between speed of light, its wavelength and its frequency.

$$\begin{aligned} \lambda &= \frac{c}{f} \qquad (7) \\ &= \frac{3 \times 10^8}{10^8} \\ &= 3 \text{ m} \\ \Rightarrow d &= \frac{\lambda}{4} = \frac{3}{4} = 0.75 \text{ m}. \end{aligned}$$

5. **Dodging the Police**

Police radar speed guns basically work by reflecting electromagnetic waves off the car's surface. Suppose you want to drive real fast on the motor way and somehow you have tracked exactly what the frequency of police speed radar is (usually it is somewhere around 20 GHz). Can you think of a design feature that you can include on the car's surface so that your car becomes invisible to the police radar.

Answer 5**:**

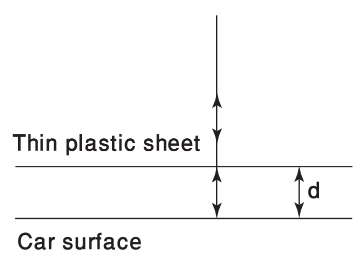
The frequency of the police speed radar is $f = 20$ GHz$= 2 \times 10^{10}$ Hz. We first find its wavelength as it will be useful later. For this purpose, we use equation (**??**).

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{2 \times 10^{10}} = 1.5 \times 10^{-2}.$$

We want to design a feature for the car surface so that the car becomes invisible for the radar. How about if we design that causes destructive interference for the waves that reflects off the surface? That should work.

Let us cover the car by a thin plastic sheet that partially reflects the incoming waves. Let this sheet be at a distance d from the surface of the car. Then we would obtain full destructive interference if the following condition holds.

$$\begin{aligned} 2d &= \frac{\lambda}{2} \\ \Rightarrow d &= \frac{\lambda}{4} = 3.75 \times 10^{-3} \text{ m}. \end{aligned}$$



6. **Rainbow on spilled oil**

Suppose there is a thin film of oil spread over some water. The film is only 4.8 micro meter thick. We look at this film perpendicularly from the top. Which color of the light will be seen brightest? Which color will be most suppressed? For simplicity, consider only three colors; Red with a wavelength of 400 nano meter, Green with a wavelength of 520 nanometer and Blue with a wavelength of 640 nano meter. Recall that the light rays will be reflected from top and bottom of the film and those two rays interfere.

Answer 6**:**

The thickness of the film is $d = 4.8\ \mu\text{m} = 4.8 \times 10^{-6}$ m. For constructive interference of reflected waves there should be an integral number of wavelengths within the film, so the condition for constructive interference is

$$2d = n\lambda,$$

where n is some positive integer.

Similarly for destructive interference, there should be odd number of half wavelengths within the film, so the condition for destructive interference is

$$ad = \left(n + \frac{1}{2}\right)\lambda$$

where n is some non-negative integer. We want to see if for light beams of wavelengths 400 nm, 520 nm and 640 nm, $2d/\lambda$ is an integer (as in the case of constructive interference) or half an integer (as in the case of destructive interference).

For $\lambda = 400$ nm:

$$\frac{2d}{\lambda} = \frac{9.6 \times 10^{-6}}{4 \times 10^{-7}} = 24,$$

which is an integer, hence red light will interfere constructively.

For $\lambda = 520$ nm:

$$\frac{2d}{\lambda} = \frac{9.6 \times 10^{-6}}{5.2 \times 10^{-7}} = 18.5,$$

which is half an integer, so green light will interfere destructively.

For $\lambda = 640$ nm:

$$\frac{2d}{\lambda} = \frac{9.6 \times 10^{-6}}{6.4 \times 10^{-7}} = 15.$$

Hence a blue light will also interfere constructively.

7. **Measuring Wavelengths with an Interferometer**

Consider the interferometer as shown in figure. Two monochromatic light rays that are initially coming from same source (and hence being initially in phase) travel the two different paths and finally we look at them on the screen and find an interference pattern. Suppose we change the position of one of the mirrors (mirror A in the figure) by 0.0125 mm and find that the bright and dark fringes change into each other 100 times during this process. What must have been the frequency of the light ray?

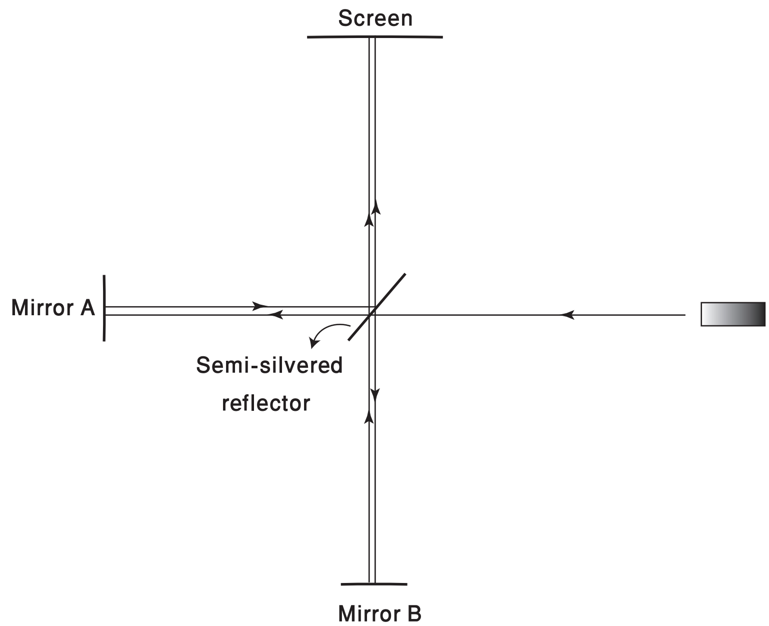


Fig.1 interferometer

Answer 7:

If we change the position of one of the mirrors by d such that $2d = \lambda/2$, then a bright fringe changes into a dark fringe, and a dark fringe changes into a bright fringe. If dark and bright fringes change into each other 100 times during the process, then the distance D by which mirror has been displaced must be 100 times d. So,

$$D = 100d = 100 \times \frac{\lambda}{4} = 25\lambda \tag{8}$$

We are given the value of D in the question, $D = 0.0125$ mm $= 1.25 \times 10^{-5}$. Then we can find the wavelength of light using equation (**??**).

$$\lambda = \frac{D}{25} = \frac{1.25 \times 10^{-5}}{25} = 5 \times 10^{-7} \text{ m} = 500 \text{ nm}.$$

Now using equation (**??**), we can find the frequency of light wave.

$$f = \frac{c}{\lambda} = \frac{3 \times 10^{8}}{5 \times 10^{-7}} = 6 \times 10^{14} \text{ Hz}.$$

8. **Measuring Gravitational Waves**

Think of the LIGO interferometer and suppose a gravitational wave passes through it. The gravitational wave has a direction and contracts one of the arm by a fraction

x while lengthens the other arm by same fraction x for a short while. While this happens the carefully analyzed interferometer registers that a bright fringe has changed into a dark fringe. Supposing that the Laser being used had a wavelength of 700 nm and each arm has a length of 4 km, what is the fractional change x in lengths that the gravitational wave must have caused on its path as it passes through. The interferometer can actually detect even a billionth part of a full shift from bright to dark fringe. What is its best sensitivity in terms of fractional length change that it can measure using this laser beam?

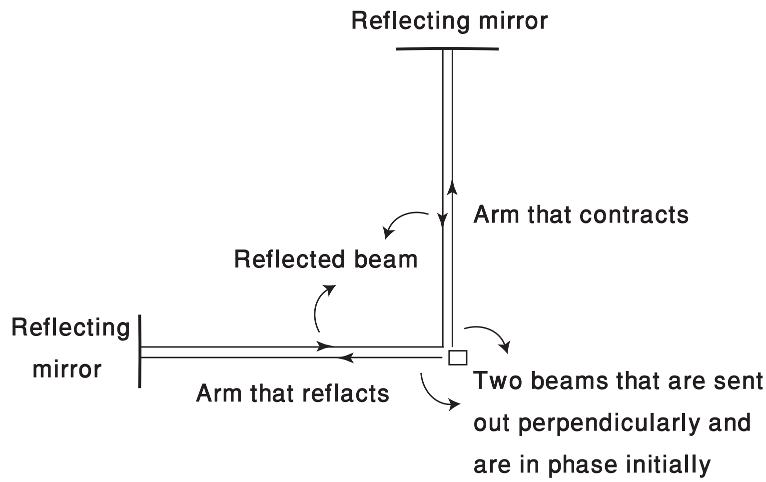


Fig.2 LIGO setup

Answer 8**:**

The gravitational wave under consideration extends one arm of LIGO interferometer and contracts its other arm by same fraction x. This means that x is the ratio of change in the length of an arm, say $\Delta\ell$, to the initial length of the arm, say L, which is 4 km.

$$\begin{aligned} x &= \frac{\Delta\ell}{L} \\ &= \frac{\Delta\ell}{4 \times 10^3}. \end{aligned}$$

We want to find x given that the gravitational wave causes a bright fringe to go into a dark fringe. Since $x = \Delta\ell/(4 \times 10^3)$, finding $\Delta\ell$ would suffice, so let us find the value of $\Delta\ell$.

As one of the arms of the interferometer contracts by $\Delta\ell$, and other arm of the interferometer expands by $\Delta\ell$, the total difference in the optical path length of light would be $4\Delta\ell$. For a bright fringe to go into a dark fringe, the distance $4\Delta\ell$ must contain half a wavelength.

$$\begin{aligned} 4\Delta\ell &= \frac{\lambda}{2} \\ \Rightarrow \Delta\ell &= \frac{\lambda}{8}. \end{aligned}$$

We are given that wavelength of light is 700 nm, so

$$\Delta\ell = \frac{7 \times 10^{-7}}{8} = 8.75 \times 10^{-8} \text{ m}.$$

We can now find the fractional change in length.

$$x = \frac{\Delta\ell}{L} = \frac{8.75 \times 10^{-8}}{4 \times 10^{3}} = 2.2 \times 10^{-11}.$$

Now, it is given that the interferometer can even detect a billionth part $(1/10^9)$ of this value of x. So the sensativity of LIGO interferometer in terms of fractional length change is of the order of $\frac{10^{-11}}{10^9} = 10^{-20}$.