Problem Set 3: Bohr's Atom

Solution

1. A photon is emitted as an atom makes a transition from $n = 4$ to $n = 2$ level. What is the frequency, wavelength and energy of the emitted photon?

   Answer 1

   we are given that,

$$\text{Initial orbit of electron} = n_i = 4$$
$$\text{Final orbit of electron} = n_f = 2.$$

   Also we know that,

$$\text{Speed of electromagnetic waves} = c = 3 \times 10^8 \text{ m/s}$$
$$\text{Planck's constant} = h = 6.63 \times 10^{-34} \text{ Js}.$$

   We want to calculate the frequency, wavelength and energy of the emitted photon, when the atom makes a transition from $n_i = 4$ to $n_f = 2$.

   Frequency of the emitted photon can be calculated by using the equation,

$$E = hf, \quad \Rightarrow f = \frac{E}{h}.$$

   Let's at first calculate E by using the following relation,

$$\begin{aligned} E &= -13.6\left[\frac{1}{n_f^2} - \frac{1}{n_i^2}\right] \text{ eV} \\ &= -13.6\left[\frac{1}{2^2} - \frac{1}{4^2}\right] \text{ eV} \\ &= -13.6\left[\frac{1}{4} - \frac{1}{16}\right] \text{ eV} \\ &= -13.6 \times \frac{3}{16} \text{ eV} \\ &= 2.55 \text{ eV}. \end{aligned}$$

   Thus frequency of photon will be,

$$\begin{aligned} f &= \frac{E}{h} = \frac{2.55 \text{ eV}}{6.63 \times 10^{-34} \text{ Js}} \\ &= \frac{2.55 \times 1.6 \times 10^{-19} \text{ J}}{6.63 \times 10^{-34} \text{ Js}} \\ &= 6.15 \times 10^{14} \text{ Hz}. \end{aligned}$$

Wavelength of the emitted photon can be calculated by using the following equation.

$$\begin{aligned} c &= f\lambda \\ \Rightarrow \lambda &= \frac{c}{f} = \frac{3 \times 10^8 \text{ m/s}}{6.15 \times 10^{14} \text{ Hz}} \\ &= 4.875 \times 10^{-7} \text{ m} \\ &= 488 \text{ nm}. \end{aligned}$$

2. For the Balmer series i.e., the atomic transitions where final state of the electron is $n = 2$, what is the longest and shortest wavelength possible? Is any of the frequency of Lyman series, which corresponds to transitions where electron ends up in $n = 1$ level, in the visible region?

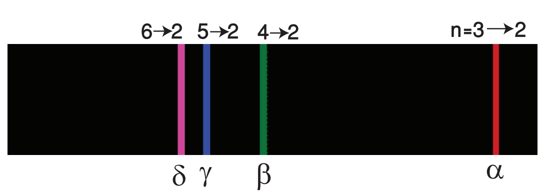
Answer 2

The atomic transitions where final state of the electron is $n = 2$, atoms emit a series of lines in the visible part of the spectrum. This series is called the Balmer Series. Balmer examined the four visible lines in the spectrum of the hydrogen atom. Wave length of this series can be calculated by using the equation,

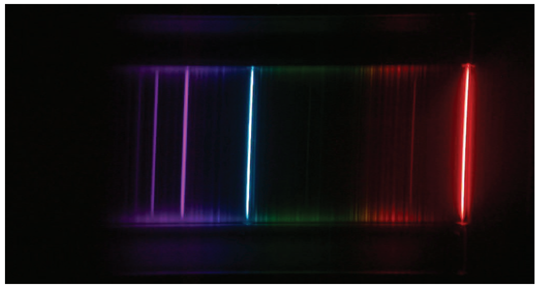
$$\frac{1}{\lambda} = R\left[\frac{1}{2^2} - \frac{1}{n^2}\right], \qquad \text{where } n = 3, 4, 5, ...$$

R is the Rydberg constant, whose value is $1.097 \times 10^7 \text{ m}^{-1}$. The number n is just an integer; the above formula gives the longest wavelength, when $n = 3$, and gives each of the shorter wavelengths as n increases. From Balmer's equation, it looks like when n gets bigger, the lines should start getting really close together. That's exactly right; as n gets larger, 1 over n squared gets smaller, so there's less and less difference between the consecutive lines. We can see that the series has a limit, that is, as n gets larger and larger, the wavelength gets closer and closer to one particular value. If n is infinity, then 1 over n squared is 0, and the corresponding wavelength is shortest.

The individual lines in the Balmer series are given the names Alpha, Beta, Gamma, and Delta, and each corresponds to a n_i value of 3, 4, 5, and 6 respectively as shown in the figure below.



A picture from experimental demonstration is shown below.



The longest wavelength corresponds to the smallest energy difference between energy levels, which in this case will be between $n = 2$ and $n = 3$.

Wavelength for transition from $n = 3$

$$\begin{aligned}
\frac{1}{\lambda} &= R\left[\frac{1}{2^2} - \frac{1}{n^2}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \left[\frac{1}{2^2} - \frac{1}{3^2}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \left[\frac{1}{4} - \frac{1}{9}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \frac{5}{36} \\
&= 1.524 \times 10^6 \text{ m}^{-1} \\
\Rightarrow \lambda &= 6.56 \times 10^{-7} \text{ m} = 656 \text{ nm}.
\end{aligned}$$

Thus the longest wavelength in the Balmer series is 656 nm.

The shortest wavelength corresponds to the largest energy difference between energy levels, which in this case will be between $n = 2$ and $n = \infty$.

Wavelength for transition from $n = \infty$

$$\begin{aligned}
\frac{1}{\lambda} &= R\left[\frac{1}{2^2} - \frac{1}{n^2}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \left[\frac{1}{2^2} - \frac{1}{\infty^2}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \left[\frac{1}{4} - \frac{1}{\infty}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \frac{1}{4} \\
&= 2.74 \times 10^6 \text{ m}^{-1} \\
\Rightarrow \lambda &= 3.65 \times 10^{-7} \text{ m} = 365 \text{ nm}.
\end{aligned}$$

Thus the shortest wavelength in the Balmer series is 365 nm.

In order to see that whether any of the frequency of Lymen series is in the visible region, let's calculate the largest and smallest frequencies of Laymen series. the wavelength of the Laymen series is given by,

$$\frac{1}{\lambda} = R\left[1 - \frac{1}{n^2}\right], \qquad \text{where } n = 2, 3, 4, 5, ...$$

The longest wavelength corresponds to the smallest energy difference between energy levels, which in this case will be between $n = 1$ and $n = 2$.

Wavelength for transition from $n = 2$

$$\begin{aligned}
\frac{1}{\lambda} &= R\left[1 - \frac{1}{n^2}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \left[1 - \frac{1}{2^2}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \left[1 - \frac{1}{4}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \frac{3}{4} \\
&= 8.23 \times 10^6 \text{ m}^{-1} \\
\Rightarrow \lambda &= 1.22 \times 10^{-7} \text{ m} = 122 \text{ nm}.
\end{aligned}$$

Thus the longest wavelength in the Laymen series is 122 nm, and corresponding frequency will be,

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8 \text{ m/s}}{1.22 \times 10^{-7} \text{ m}} = 24.7 \times 10^{14} \text{ Hz}.$$

The shortest wavelength corresponds to the largest energy difference between energy levels, which in this case will be between $n = 1$ and $n = \infty$.

Wavelength for transition from $n = \infty$

$$\begin{aligned}
\frac{1}{\lambda} &= R\left[1 - \frac{1}{n^2}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \left[1 - \frac{1}{\infty^2}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times \left[1 - \frac{1}{\infty}\right] \\
&= 1.097 \times 10^7 \text{ m}^{-1} \times 1 \\
&= 1.097 \times 10^7 \text{ m}^{-1} \\
\Rightarrow \lambda &= 9.12 \times 10^{-8} \text{ m} = 91.2 \text{ nm}.
\end{aligned}$$

Thus the shortest wavelength in the Laymen series is 91.2 nm, and corresponding frequency will be,

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8 \text{ m/s}}{9.12 \times 10^{-8} \text{ m}} = 32.9 \times 10^{14} \text{ Hz}.$$

From above we can see that frequency range of Laymen series is 24.7×10^{14} Hz to 32.9×10^{14}. As frequency range of visible light is from 4×10^{14} to 8×10^{14}, therefore no frequency of Laymen series is in the visible region.

3. A Hydrogen atom initially in its ground state i.e., $n = 1$ level, absorbs a photon and ends up in $n = 4$ level. What must have been the frequency of the photon? Now the electron makes spontaneous emission and comes back to the ground state. What are the possible frequencies of the photons emitted during this process.

Answer 3

We are given that,

$$\begin{aligned} \text{Initial state of electron} &= n_i = 1 \\ \text{final state of electron} &= n_f = 4. \end{aligned}$$

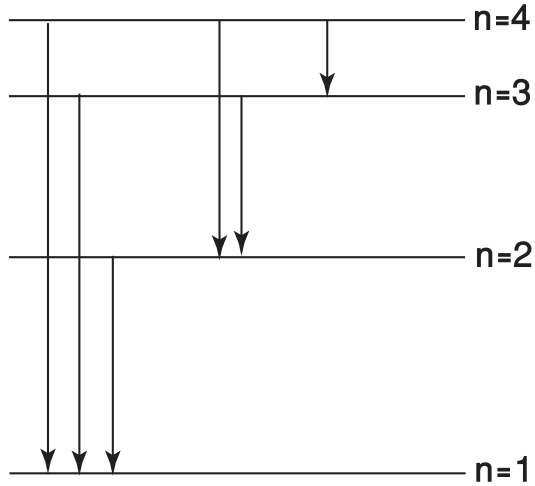
We want to calculate the frequency of photon absorbed by the electron to make this transition. As we know that energy of photon absorbed is given by,

$$\begin{aligned} E &= -13.6\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \text{ eV} \\ hf &= -13.6\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \text{ eV} \\ f &= -\frac{13.6}{h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \text{ eV} \\ &= -\frac{13.6}{4.14 \times 10^{-15} \text{ eVs}}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \text{ eV} \\ &= -3.28 \times 10^{15}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \text{ Hz} \\ f_{1\to4} &= -3.28 \times 10^{15}\left(\frac{1}{4^2} - \frac{1}{1^2}\right) \text{ Hz} \\ &= -3.28 \times 10^{15}\left(\frac{1}{16} - 1\right) \text{ Hz} \\ &= -3.28 \times 10^{15} \times \frac{-15}{16} \text{ Hz} \\ &= +3.08 \times 10^{15} \text{ Hz}, \end{aligned}$$

where positive sign shows that photon has been absorbed.

Spontaneous emission

For spontaneous emission electron can have transitions as shown in the figure below.



Direct Transition from $n = 4$ to $n = 1$

If electron jumps directly from level 4 to level 1, the frequency of the emitted photon will be equal to the frequency of the absorbed photon when it had jumped from level 1 to level 4, therefore,

$$f_{4\to 1} = -3.08 \times 10^{15} \text{ Hz},$$

where negative sign shows that energy has been emitted.

Indirect Transitions

Indirectly the atom can make the following transitions to fall down from level 4 to level 1.

(a) From level 4 to level 2 and then from 2 to level 1.

(b) From level 4 to level 3 and then from 3 to level 1.

(c) From level 4 to level 3, then from level 3 to level 2 and then finally from level 2 to level 1.

Let's calculate frequencies for all these transitions one by one.

From $n = 4$ **to** $n = 2$

$$\begin{aligned} f_{4\to 2} &= -3.28 \times 10^{15} \left(\frac{1}{2^2} - \frac{1}{4^2} \right) \text{ Hz} \\ &= -6.15 \times 10^{14} \text{ Hz.} \end{aligned}$$

From $n = 2$ **to** $n = 1$

$$\begin{aligned} f_{2\to 1} &= -3.28 \times 10^{15} \left(\frac{1}{1^2} - \frac{1}{2^2} \right) \text{ Hz} \\ &= -2.46 \times 10^{15} \text{ Hz.} \end{aligned}$$

From $n = 4$ **to** $n = 3$

$$\begin{aligned} f_{4\to 3} &= -3.28 \times 10^{15} \left(\frac{1}{3^2} - \frac{1}{4^2} \right) \text{ Hz} \\ &= -1.6 \times 10^{14} \text{ Hz.} \end{aligned}$$

From $n = 3$ **to** $n = 1$

$$\begin{aligned} f_{3\to 1} &= -3.28 \times 10^{15} \left(\frac{1}{1^2} - \frac{1}{3^2} \right) \text{ Hz} \\ &= -2.92 \times 10^{15} \text{ Hz.} \end{aligned}$$

From $n = 3$ **to** $n = 2$

$$\begin{aligned} f_{3\to 2} &= -3.28 \times 10^{15} \left(\frac{1}{2^2} - \frac{1}{3^2} \right) \text{ Hz} \\ &= -4.57 \times 10^{1} \text{ Hz.} \end{aligned}$$

Negative sign with each frequency represents that photon has been emitted.

4. **Non-relativistic Nature of Electron Inside Bohr's Atom**

Using Bohrs quantization rule, derive a formula for electrons speed in the quantized Bohrs orbits. By putting in the values of constants, explicitly derive the value of electrons speed in $n = 1$ orbit. What fraction of speed of light it is? Is it justified to treat electrons motion as non-relativistic in Hydrogen atom as we have been doing? What happens to the speed in higher levels?

Answer 4

The basic feature of quantum mechanics that is incorporated in the Bohr Model the energy of the particles in the Bohr atom is restricted to certain discrete values. One says that the energy is quantized. This means that only certain orbits with certain radii are allowed; orbits in between simply don't exist.

In the Bohr model, the wavelength associated with the electron is given by the De-Broglie relationship,

$$\lambda = \frac{h}{mv},$$

and the standing wave condition that circumference = whole number of wavelengths. In the hydrogenic case, the number n is the principal quantum number.

$$2\pi r = n\lambda.$$

These can be combined to get an expression for the angular momentum of the electron in orbit. (Note that this assumes a circular orbit, a generally unwarranted assumption.)

$$\begin{aligned} \frac{2\pi r}{n} &= \frac{h}{mv} \\ mvr &= \frac{nh}{2\pi} \\ mvr &= n\hbar \\ r &= \frac{n\hbar}{mv}. \end{aligned} \tag{1}$$

The electron is held in a circular orbit by electrostatic attraction. The centripetal force is equal to the Coulomb force.

$$\begin{aligned} \frac{mv^2}{r} &= \frac{Zke^2}{r^2} \\ r &= \frac{Zke^2}{mv^2}. \end{aligned} \tag{2}$$

Compare equation (**??**) and (**??**) we get,

$$\begin{aligned} \frac{n\hbar}{mv} &= \frac{Zke^2}{mv^2} \\ v &= \frac{Zke^2}{n\hbar}. \end{aligned} \tag{3}$$

where

$$\begin{aligned} Z &= \text{Atomic number for Hydrogen} = 1 \\ k &= \text{Coulomb's constant} = 9 \times 10^9 \text{ Nm}^2/\text{C}^2 \\ e &= \text{Charge on an electron} = 1.6 \times 10^{-19} \text{ C} \\ \hbar &= \text{Reduced Planck's constant} = 1.05 \times 10^{-34} \text{ J.s} \end{aligned}$$

By putting in the values of constants, electron's speed will be,

$$\begin{aligned} v &= \frac{Zke^2}{n\hbar} \\ &= \frac{1 \times 9 \times 10^9 \text{ Nm}^2/\text{C}^2 \times (1.6 \times 10^{-19} \text{ C})^2}{1 \times 1.05 \times 10^{-34} \text{ J.s}} \\ &= 2.19 \times 10^6 \text{ m/s.} \end{aligned}$$

This speed of electron can be compared to the speed of light by taking ratio to these two speeds.

$$\text{fraction} = \frac{v}{c} = \frac{2.19 \times 10^6 \text{ m/s}}{3 \times 10^8 \text{ m/s}} = 7.3 \times 10^{-3}.$$

This is justified to treat this electron's motion as non-relativistic because the speed of electron is not comparable to the speed of light rather it is pretty small.

From equation (**??**) we can see that $v \propto \frac{1}{n}$, therefore v decreases as n increases.

5. **An Atom with Anti-Electron in the Center**

Go back to the derivation of Bohrs formula for quantized orbits. In reality, the electron is not merely moving around the proton, rather the two particles are moving around the center of mass and we can reduce the problem to that of a single particle (of reduced mass) moving around this center of mass. This is what you did in Mechanics course to solve two body central force problem. Explain why did then we put mass of electron in all our formulas instead of reduced mass. Positronium is an atom made up of a positive electron (a positron, the anti-particle of electron) and a usual negative electron moving around each other. For this atom, find the Bohrs radius and first few frequencies of "Balmer" series, i.e. when this atom makes transitions to $n = 2$ orbit from $n = 3, 4, 5$ orbits. Compare the frequencies with Hydrogen atom.

Answer 5

The reduced mass is the "effective" inertial mass appearing in the two-body problem of Newtonian mechanics. It is a quantity which allows the two-body problem to be solved as if it were a one-body problem.

In case of Hydrogen atom, electron and proton are moving around the center of mass. As we know that the mass of proton is very large as compare to the mass of electron, therefore center of mass shifted nearer to the proton. Infact proton is at the position of the center of mass and may be considered as stationary, while electron moving around it. Reduced mass for such a system can be calculated by using the equation,

$$\text{Reduced mass } = \mu = \frac{m_e m_p}{m_e + m_p},$$

where $m_p >> m_e, \;\; \Rightarrow m_e + m_p \approx m_p$. Thus reduced mass will become,

$$\begin{aligned}
\mu &= \frac{m_e m_p}{m_e + m_p} \\
\mu &\approx \frac{m_e m_p}{m_p} \\
\mu &\approx m_e.
\end{aligned}$$

From above we can see that the reduced mass is approximately equal to the mass of electron, that is why in our formulas we put mass of electron instead of reduced mass.

Now let's consider the positronium. Positronium is made up of a positron (anti particle of electron) and an electron. Since electron and positron both have the same masses the center of mass will be at the center of the line joining the two particles. In this case the reduced mass is given by,

$$\begin{aligned}
\mu &= \frac{(m_{e^-})(m_{e^+})}{(m_{e^-}) + (m_{e^+})} \\
\mu &= \frac{(m_{e^-})(m_{e^+})}{2(m_{e^+})} \\
\mu &= \frac{m_{e^-}}{2} = \frac{m_e}{2}.
\end{aligned}$$

Bohr radius is given by,

$$r = \frac{n^2 \hbar^2}{k m_e e^2}.$$

For positronium Bohr radius will be,

$$\begin{aligned}
r &= \frac{n^2 \hbar^2}{k \mu e^2} \\
&= \frac{2 n^2 \hbar^2}{k m_e e^2},
\end{aligned}$$

Substitute values of constants, we get,

$$\begin{aligned}
r &= \frac{2 \times (1.05 \times 10^{-34} \text{ J.s})^2}{9 \times 10^9 \text{ Nm}^2/\text{C}^2 \times 9.1 \times 10^{-31} \text{ kg} \times (1.6 \times 10^{-19} \text{ C})^2} \\
&= 1.05 \times 10^{-10} \text{ m} \\
&= 1.05\text{Å}.
\end{aligned}$$

Numerical value for Bohr's radius for hydrogen atom is 0.529 Å. Therefore Bohr's radius for positronium is increased by 2 times. As we know that Rydberg constant in case of hydrogen atom is given by,

$$R = \frac{m_e e^4}{8\varepsilon_0^2 h^2}.$$

For positronium Rydberg constant will be,

$$\begin{aligned} R' &= \frac{\mu e^4}{8\varepsilon_0^2 h^2} \\ R' &= \frac{m_e e^4}{2 \times 8\varepsilon_0^2 h^2} \\ &= \frac{R}{2}. \end{aligned}$$

Since Rydberg constant for positronium is reduced to half, formula for Balmer series for positronium is given by,

$$\begin{aligned} \frac{1}{\lambda} &= \frac{R}{2}\left[\frac{1}{2^2} - \frac{1}{n^2}\right], \qquad \text{where } n = 3, 4, 5, ... \\ \frac{f}{c} &= \frac{R}{2}\left[\frac{1}{2^2} - \frac{1}{n^2}\right] \\ f &= \frac{cR}{2}\left[\frac{1}{2^2} - \frac{1}{n^2}\right]. \end{aligned}$$

For n=3

$$\begin{aligned} f_1' &= \frac{3 \times 10^8 \text{ m/s} \times 1.097 \times 10^7 \text{ m}^{-1}}{2}\left[\frac{1}{2^2} - \frac{1}{3^2}\right] \\ &= \frac{3 \times 10^8 \text{ m/s} \times 1.097 \times 10^7 \text{ m}^{-1}}{2} \times \frac{5}{36} \\ &= 2.28 \times 10^{14} \text{ Hz}. \end{aligned}$$

For n=4

$$\begin{aligned} f_2' &= \frac{3 \times 10^8 \text{ m/s} \times 1.097 \times 10^7 \text{ m}^{-1}}{2}\left[\frac{1}{2^2} - \frac{1}{4^2}\right] \\ &= \frac{3 \times 10^8 \text{ m/s} \times 1.097 \times 10^7 \text{ m}^{-1}}{2} \times \frac{3}{16} \\ &= 3.08 \times 10^{14} \text{ Hz}. \end{aligned}$$

For n=5

$$\begin{aligned} f_3' &= \frac{3 \times 10^8 \text{ m/s} \times 1.097 \times 10^7 \text{ m}^{-1}}{2}\left[\frac{1}{2^2} - \frac{1}{5^2}\right] \\ &= \frac{3 \times 10^8 \text{ m/s} \times 1.097 \times 10^7 \text{ m}^{-1}}{2} \times \frac{21}{100} \\ &= 3.46 \times 10^{14} \text{ Hz}. \end{aligned}$$

For Hydrogen atom frequencies are calculated as below.

$$\begin{aligned} f_1 &= 2f_1' = 2 \times 2.28 \times 10^{14} \text{ Hz} = 4.57 \times 10^{14} \text{ Hz} \\ f_2 &= 2f_2' = 2 \times 3.08 \times 10^{14} \text{ Hz} = 6.17 \times 10^{14} \text{ Hz} \\ f_3 &= 2f_3' = 2 \times 3.46 \times 10^{14} \text{ Hz} = 6.92 \times 10^{14} \text{ Hz}. \end{aligned}$$

6. **Patching Bohr's Model with Larger Everyday World**

We saw that in class how we can mesh up the fact of photon nature of light, that interaction of light with matter is always in terms of a discrete bundles of energy interacting with matter, with the smooth flow of radiation we are used to of seeing in everyday life, and which is characteristic of Maxwells theory, when the total energies involved are so large that we tend to ignore the small discrete packets. Let us see how this works out in case of Hydrogen atom. Just like an electron oscillating with a given frequency emits radiations of same frequency, Maxwells theory predicts that an electron orbiting in a circle with a certain frequency of revolution will emit light of the same frequency. As it does so, it loses energy and its orbit shortens, increasing the frequency of revolution and in next instant it emits radiation of a bit larger frequency in a continues fashion. Since we can test all of this for energies of larger scales, this picture should come out from our quantum formulas in those limits.

(a) Show that in Bohrs theory, the frequency of photon emitted as the electron makes a transition from an orbit with quantum number n to an orbit of quantum number $n-1$ is given by,

$$f = \frac{m_e e^4}{8\epsilon_o^2 h^3}\left[\frac{2n-1}{(n-1)^2 n^2}\right]$$

For large orbits, as we encounter in everyday situations, n is very large. Take that limit in the above formula, ignoring additions of order 1 to the large number n.

(b) Now calculate frequency of revolution of the electron moving in an orbit quantized according to Bohrs formula (put Bohrs formula for electrons radius in n-th orbit) and show that it comes out exactly equal to the frequency of photon derived in part a.

(c) Explain in detail what would one observe (about the radiation) if an electron were revolving around a proton in a macroscopic sized radius and how it matches with Maxwells predictions. Do we need to use quantum theory for motion of particles in macroscopic orbits?

(d) Find the frequency of photons emitted by an electron moving around a proton in a radius of 1 cm.

Answer 6

(a) As we have already discussed in lecture 9 that total energy of an electron in nth orbit is given by,

$$\begin{aligned} E_n &= K.E. + P.E. \\ &= \frac{1}{2}mv^2 - \frac{Ze^2}{kr} \qquad (4) \\ \text{Coulomb's Force} &= \text{Centripetal Force} \\ \frac{Ze^2}{4\pi\varepsilon_0 r^2} &= \frac{mv^2}{r} \\ \Rightarrow \frac{Ze^2}{4\pi\varepsilon_0 r} &= mv^2 \\ \frac{Ze^2}{8\pi\varepsilon_0 r} &= \frac{mv^2}{2}. \end{aligned}$$

Thus equation (**??**) can be written as,

$$\begin{aligned} E_n &= \frac{Ze^2}{8\pi\varepsilon_0 r} - \frac{Ze^2}{4\pi\varepsilon_0 r} \\ &= -\frac{Ze^2}{8\pi\varepsilon_0 r}, \end{aligned}$$

where r is the Bohr radius that is given by,

$$r = \frac{\varepsilon_0 n^2 h^2}{\pi Z m_e e^2}.$$

Therefore total energy of nth orbit will become,

$$\begin{aligned} E_n &= -\frac{Ze^2}{8\pi\varepsilon_0} \times \frac{\pi Z m_e e^2}{\varepsilon_0 n^2 h^2} \\ &= -\frac{m_e Z^2 e^4}{8n^2 \varepsilon_0^2 h^2}. \end{aligned}$$

Similarly total energy for $(n-1)$th orbit will be,

$$E_{(n-1)} = -\frac{m_e Z^2 e^4}{8(n-1)^2 \varepsilon_0^2 h^2}.$$

Energy of photon emitted as the electron makes a transition from an orbit with quantum number n to an orbit of quantum number $(n-1)$ is given by,

$$\begin{aligned} E_n - E_{n-1} &= \frac{m_e Z^2 e^4}{8(n-1)^2\varepsilon_0^2 h^2} - \frac{m_e Z^2 e^4}{8n^2\varepsilon_0^2 h^2} \\ \Delta E &= \frac{m_e Z^2 e^4}{8\varepsilon_0^2 h^2}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right] \\ hf &= \frac{m_e e^4}{8\varepsilon_0^2 h^2}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right] \qquad \text{since} \;\; Z = 1 \\ f &= \frac{m_e e^4}{8\varepsilon_0^2 h^3}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right] \\ &= \frac{m_e e^4}{8\varepsilon_0^2 h^3}\left[\frac{2n-1}{n^2(n-1)^2}\right]. \end{aligned}$$

Hence proved.

(b) According to Bohr's quantization,

$$\begin{aligned} m_e v r &= n\hbar \\ m_e \times r\omega \times r &= n\hbar \qquad \text{since} \;\; v = r\omega, \end{aligned}$$

Also $\omega = 2\pi f$, $\hbar = h/2\pi$ and Bohr radius $r = \varepsilon_0 n^2 h^2/\pi Z m_e e^2$, therefore

$$\begin{aligned} m_e \times \left(\frac{\varepsilon_0 n^2 h^2}{\pi Z m_e e^2}\right)^2 \times 2\pi f &= n \times \frac{h}{2\pi} \\ f &= \frac{Z^2 m_e e^4}{4\varepsilon_0^2 n^3 h^3} \\ &= \frac{m_e e^4}{4\varepsilon_0^2 n^3 h^3} \qquad \text{since} \;\; Z = 1. \end{aligned}$$

(c) One would observe a continuous radiation because the levels would be placed so close, that distinguishing of each packet of radiation from another would be impossible.

No, we do not need quantum theory. The electron would still emit discrete photons but they are not detectable due to closely packed energy levels as $n \to \infty$.

(d) Now we are given that,

$$\text{Radius of electron's orbit} = r = 1 \text{ cm} = 10^{-2} \text{ m}.$$

We want to calculate the frequency of the photon emitted. As we have already calculated in part (b), frequency of emitted photon is given by,

$$f = \frac{m_e e^4}{4\varepsilon_0^2 n^3 h^3}.$$

Let's at first calculate n by using the equation for Bohr's radius,

$$r = \frac{n^2\hbar^2}{km_e e^2} = n^2 a_0,$$

where $a_0 = \hbar^2/km_e e^2 = 0.529\text{Å} = 5.29 \times 10^{-11}$ m, thus

$$\begin{aligned} r &= n^2 \times 5.29 \times 10^{-11} \text{ m} \\ n^2 &= \frac{r}{5.29 \times 10^{-11} \text{ m}} \\ &= \frac{10^{-2} \text{ m}}{5.29 \times 10^{-11} \text{ m}} \\ &= 1.9 \times 10^8 \\ n &= 1.37 \times 10^4. \end{aligned}$$

Thus frequency of the emitted photon will be,

$$\begin{aligned} f &= \frac{9.1 \times 10^{-31} \times (1.6 \times 10^{-19})^4}{4 \times (8.85 \times 10^{-12})^2 (1.37 \times 10^4)^3 (6.63 \times 10^{-34} \text{ J.s})^3} \\ &= 2.54 \times 10^3 \text{ Hz} \\ &= 2.54 \text{ kHz}. \end{aligned}$$

7. **A Quantum Nano Solar Cell**

A nano-scale P-N junction has only 100 atoms in its depletion region with each capable of producing only one electron-hole pair. In other words there are only 100 electrons available capable of jumping from valance level (band) to the conduction level. To start, all the electrons are in valence level and hence there is no free electron or hole so no conduction occurs. When a light of suitable frequency and a certain given intensity is shone on this junction continuously, it is found that there are 15 electrons promoted to conduction level at any given time, which are then swept away by the internal field producing a current.

(a) Recalling the facts which we discussed in class regarding the interaction of light with electrons, argue that if we double the intensity, the number of free electrons (electrons in conduction level) will not be doubled but would be less than that.

(b) Show that no matter how much light we shine, we will never be able to promote more than 50 electrons to conduction level in a steady state.

Answer 7

(a) Consider 0K temperature (just to eliminate any discrepancies about value of temperature). At any given time (under intensity of light), there are 15 electrons in conduction level. This means that there is a dynamic equilibrium established of let's say 20 electrons going into conduction level and 5 dropping back to valence band. Probability of an electron absorbing a photon and going to conduction band is same as that of it emitting a photon and returning to valence band from conduction band. Thus when we double the intensity there is a probability of electrons jumping back to valence band while initially there was not as there were no electrons in conduction band. So doubling intensity would result a decrease of electrons in conduction band.

(b) When we manage to send 50 electrons in conduction shell, now no more electrons can jump to conduction band as that many would jump back to valence band. Hence number of electrons from valence to conduction shell is equal to number of electrons coming from conduction to valence shell.

8. **Laser Characteristics**

Recall our laser set-up from the class with energy levels 1, 2 and 3. Level 2 is the highest and Level 3 has energy in between levels 1 and 2. Level 2 is very short lived and spontaneously goes down to level 3 with much more probability than going to level 1. Level 3 on the other hand is long lived and obviously electron there can only go to level 1 by spontaneous emission. Photons of energy $E_{12} = E_2 - E_1$ are used to pump electrons and achieve population inversion between level 1 and 3.

For each of the following changes, describe what would change in output laser characteristics in terms of its frequency and power and what changes would be required in the light that is being used for pumping in terms of its frequency and power.

(a) Both levels 2 and 3 are raised by same fixed amount with respect to level 1. All probabilities remaining same.

(b) Level 2 remains same while level 3 is brought closer to the level 1, all probabilities remaining same.

(c) Level 3 remains same while level 2 is raised up with respect to level 1, all probabilities remaining same.

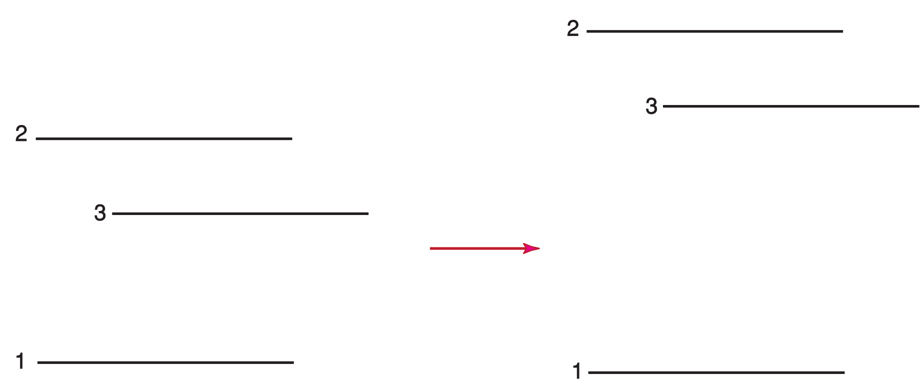
(d) All levels stay at the same level but the probability of level 2 going to level 3 by spontaneous emission is reduced (but still higher) compared to it going to level 1.

(e) All else remains same but the probability of level 2 going to 1 by spontaneous emission becomes more than it going to level 3.

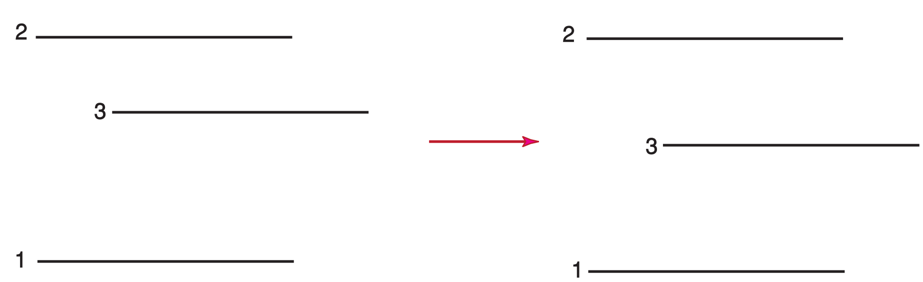
(f) All else remains same but life time of level 3 is reduced.

Answer 8

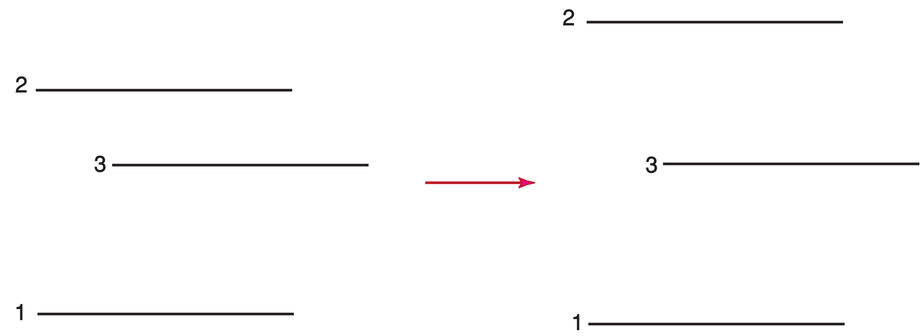
(a) The photons of energy $E_2 - E_1 = E_{12}$ would be required to achieve population inversion. As difference increased, thus pumping photon frequency and hence power would be increased. Also as $E_{31} = E_3 - E_1$ is increased, the output laser photons frequency and hence power is increased.



(b) The frequency and power of input light remains the same as E_{12} is unchanged. The frequency and hence power of output laser light is decreased.



(c) As $E_2 - E_1 = E_{12}$ has increased, thus pumping photon's frequency and hence power will increase. The frequency and hence power of output laser light will remain the same, because the energy difference of level 3 and 1 remains the same.



(d) As probability has reduced, the lifetime of level 2 has increased. this means that it would longer time for photons to go from level 2 to 3 and then to 1. Hence at a time less number of photons will be emitted in output laser, so power would decrease but frequency of output laser remains the same. Input light remain the same.

(e) The output laser no longer be a coherent source of light as now photons would jump back from level 2 to level 1 at same probability and population inversion would not be achieved. So no laser would be seen at output.

(f) This means that electrons probability of jumping from 2 to 1 increased, thus large number of photons would jump to 1, hence output laser power would remain the same.

9. Pair creation is a phenomenon in which a photon is converted to a pair of particle and antiparticle, both of exactly same mass. On the other hand in pair annihilation, the particle anti-particle pair annihilate each other, are vanished, and their energy is converted into giving off a photon.

   (a) Consider a photon of angular frequency ω which is converted to a particle anti-particle pair at a height h above the ground. The particles are produced in rest with having no energy due to motion. What would be the mass of each particle in terms of ω?

   (b) Now the two particles fall down the height h. What is the total energy possessed by the particles downstairs? Down there, they annihilate each other and their energy is converted to give off a photon. What is the frequency of this photon, in terms of ω and h?

(c) Now we send this newly created photon back to height h. In order to avoid the creation of energy, this photon must lose energy so that at height h it just have enough energy to create the pair of same mass again, without having any extra energy. What must be this energy loss for this to happen? What is the change in frequency in climbing up the height h? Does it agree with the formula we derived in class?

Answer 9

(a) As the photon has to be created two particles of equal mass and they have no energy due to motion implying their velocity is zero. Thus according to law of conservation of energy,

$$
\begin{aligned}
E_{\text{photon}} &= E^+E^- \\
\hbar\omega &= m_0c^2 + m_0c^2 \\
\hbar\omega &= 2m_0c^2 \\
\Rightarrow m_0 &= \frac{\hbar\omega}{2c^2},
\end{aligned}
$$

where m_0 is the mass of each particle.

(b) As they fall down they would gain an energy mgh, where $m = m_0$ =rest mass. As they were at rest, the velocity with which they fall is not relativistic.

$$
\begin{aligned}
E_1 &= m_0c^2 + mgh \\
E_{\text{total}} &= 2E_1 = 2(m_0c^2 + mgh),
\end{aligned}
$$

is the total energy of the particle downstairs. Now at the time of annihilation, according to law of conservation of energy

$$\begin{aligned} E_{\text{photon}} &= 2m_0c^2 + 2m_0gh \\ \hbar\omega' &= 2(c^2+gh)m_0 \\ &= 2(c^2+gh)\frac{\hbar\omega}{2c^2} \\ \omega' &= \omega\left(1+\frac{gh}{c^2}\right) \\ 2\pi f' &= .\omega\left(1+\frac{gh}{c^2}\right) \\ f' &= \frac{\omega}{2\pi}\left(1+\frac{gh}{c^2}\right). \end{aligned}$$

Thus frequency of this photon is higher, because due to the attraction of gravity, energy of photon increase.

(c) To create pair of same mass again, the photon must have the same energy as the old photon and as it goes back to height h. Thus energy loss will be,

$$\begin{aligned} \text{Energy loss} &= \hbar\omega' - \hbar\omega \\ &= \hbar\omega\left(1+\frac{gh}{c^2}\right) - \hbar\omega \\ &= \hbar\omega + \frac{\hbar\omega gh}{c^2} - \hbar\omega \\ &= \hbar\omega\frac{gh}{c^2}. \end{aligned}$$

Change in frequency while climbing up the height will be,

$$\begin{aligned} \Delta f &= f' - f \\ &= \frac{\omega'}{2\pi} - \frac{\omega}{2\pi} \\ &= \frac{1}{2\pi}\cdot\omega\left(1+\frac{gh}{c^2}\right) - \frac{\omega}{2\pi} \\ &= \frac{\omega}{2\pi} + \frac{\omega gh}{2\pi c^2} - \frac{\omega}{2\pi} \\ &= \frac{\omega gh}{2\pi c^2} \\ &= \frac{fgh}{c^2} \end{aligned}$$

Yes it again agrees that formula we derived in class.

10. **Gravitational Red Shift**

(a) A monochromatic laser light of blue color with a wavelength of exactly 450 nm is flashed directly upwards from ground floor. Someone on the 15th floor, at a height of 50 m receives it. What wavelength they would find this light to have?

(b) A Neutron star is a very compact object with mass equal to about the mass of the sun compressed within a radius of 10 km. They are routinely observed as pulsars in the sky. Suppose the same experiment as in part a is repeated from the surface of a Neutron star. What would be the wavelength now at a height of 50 m? Would it induce a noticeable color change?

Answer 10 We are given that,

$$\begin{aligned}\text{Wavelength of blue light} &= \lambda = 450 \text{ nm} = 450 \times 10^{-9} \text{ m}\\ \text{Height of 15th floor} &= h = 50 \text{ m}.\end{aligned}$$

We want to calculate wavelength of light at 15th floor.

(a) As we know that while going up photon's angular frequency changes that can be calculated by using the equation,

$$\omega' = \omega\left(1 - \frac{gh}{2c^2}\right),$$

where $\omega = 2\pi f = \frac{2\pi c}{\lambda}$.

$$\begin{aligned}\frac{2\pi c}{\lambda'} &= \frac{2\pi c}{\lambda}\left(1 - \frac{gh}{2c^2}\right)\\ \frac{1}{\lambda'} &= \frac{1}{\lambda}\left(1 - \frac{gh}{2c^2}\right)\\ \frac{1}{\lambda'} &= \frac{1}{450 \times 10^{-9} \text{ m}}\left(1 - \frac{9.81 \text{ m/s}^2 \times 50 \text{ m}}{(3 \times 10^8 \text{ m/s})^2}\right)\\ \frac{1}{\lambda'} &= \frac{1}{450 \times 10^{-9} \text{ m}}\\ \lambda' &= 450 \times 10^{-9} \text{ m}\\ \lambda' &= 450 \text{ nm}\end{aligned}$$

So a person on 15 floor would find approximately same wavelength of light.

(b) Now we are given that,

$$\begin{aligned}\text{Mass of Neutron star} &= \text{Mass of Sun} = M_s = 2 \times 10^{30} \text{ m}\\ \text{Radius of Neutron star} &= R = 10 \text{ km} = 10^4 \text{ m}.\end{aligned}$$

Again wavelength at the height of 50 m can be calculated by using equation,

$$\frac{1}{\lambda'} = \frac{1}{\lambda}\left(1 - \frac{g'h}{c^2}\right),$$

where gravity g' of Neutron star is,

$$\begin{aligned} g' &= \frac{GM_s}{R^2} \\ &= \frac{6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2 \times 2 \times 10^{30}\ \text{m}}{(10^4\ \text{m})^2} \\ &= 1.33 \times 10^{12}\ \text{m/s}^2. \end{aligned}$$

Therefore calculations for wavelength will be as follows,

$$\begin{aligned} \frac{1}{\lambda'} &= \frac{1}{450 \times 10^{-9}\ \text{m}}\left(1 - \frac{1.33 \times 10^{12}\ \text{m/s}^2 \times 50\ \text{m}}{(3 \times 10^8\ \text{m/s})^2}\right) \\ &= \frac{0.999}{450 \times 10^{-9}\ \text{m}} \\ \lambda' &= \frac{450 \times 10^{-9}\ \text{m}}{0.999} \\ &= 450.333 \times 10^{-9}\ \text{m} \\ &= 450.333\ \text{nm}. \end{aligned}$$

Change in wavelength will be,

$$\begin{aligned} \Delta\lambda &= \lambda' - \lambda \\ &= (450.333\ \text{nm} - 450\ \text{nm}) \\ &= 0.333\ \text{nm}. \end{aligned}$$

Since wavelength change is very small a color change may not occur.