Problem Set 4: De Broglies Relations, Fourier Superpositions and Borns Interpretation

Solution

1. **Calculating de Broglie Wavelengths of Particles**

   (a) Calculate the de Broglie wavelength of an electron moving with a speed of 10^5 m/s and also that of an electron moving with a speed of 0.99×10^8 m/s. Be careful in your choice of formulae in the second case as it is relativistic.

   (b) To observe small objects, one measures the diffraction of particles whose de Broglie wavelength is comparable to objects size (or features). Find the kinetic energy in electron volts required for electrons to resolve (a) a large organic molecule of size 10 nm, (b) atomic features of size 10 nm and (c) a nucleus of size 10 fm. Repeat these calculations using alpha particles in place of electrons.

   (c) An electron and a photon both have kinetic energy equal to 50 keV . Are their de Broglie wavelengths equal? What are the values of the wavelengths for each?

   (d) Calculate the de Broglie wavelength of a proton that has been accelerated through a potential difference of 10 MV , if it started from rest.

Answer 1

(a) We are given that,

$$\begin{aligned} \text{Speed of electron} &= v = 10^5 \text{ m/s} \\ \text{de Broglie wavelength} &= \lambda = ? \\ \text{Relativistic speed of electron} &= v_{\text{rel}} = 0.99 \times 10^8 \text{ m/s} \\ \text{de Broglie wavelength} &= \lambda_{\text{rel}} = ? \end{aligned}$$

We have seen that in certain experiments electrons produce patterns attributed to waves. From this observation, we concluded that electrons have wave-like properties. We also concluded that as the energy of electrons increases their wavelength decreases. Electrons are a form of matter, so these waves are called matter waves. In quantum mechanics, the concept of matter waves was proposed by Louis de Broglie in 1924, therefore also called de Broglie waves. The concept of matter waves or de Broglie waves reflects the waveparticle duality of matter. The de Broglie relations show that the wavelength is inversely proportional to the momentum of a particle and is also called de Broglie wavelength. Also the frequency of matter waves, as deduced by de Broglie, is directly proportional to the total energy E (sum of its rest energy and the kinetic energy) of a particle.

The de Broglie equations relate the wavelength λ to the momentum p, and frequency f to the total energy E of a particle by the following relations:

$$\lambda = \frac{h}{p} \tag{1}$$

$$\text{and} \qquad f = \frac{E}{h}, \tag{2}$$

where h is the Planck's constant (named after Max Planck) and is equal to 6.63×10^{-34} Js.

Since the velocity of electron is a small fraction of speed of light in first case, we can treat it non-relativistically. We then have, $p = m_e v$, where m_e is the mass of electron which is equal to 9.1×10^{-31} kg. Thus equation (**??**) can be written as,

$$\begin{aligned} \lambda &= \frac{h}{p} = \frac{h}{m_e v} \\ &= \frac{6.63 \times 10^{-34} \text{ Js}}{9.1 \times 10^{-31} \text{ kg} \times 10^5 \text{ m/s}} \\ &= 7.28 \times 10^{-9} \text{ m} \\ &= 7.28 \text{ nm}. \end{aligned}$$

In second case the electron's speed is relativistic, so the correct formula to use for momentum is,

$$p = \frac{m_0}{\sqrt{1 - v^2/c^2}} v.$$

Thus equation (**??**) can be written as,

$$\begin{aligned} \lambda &= \frac{h}{p} = \frac{h\sqrt{1 - v^2/c^2}}{m_0 v} \\ &= \frac{6.63 \times 10^{-34} \text{ Js}\sqrt{1 - (0.99 \times 10^8 \text{ m/s})^2/(3 \times 10^8 \text{ m/s})^2}}{9.1 \times 10^{-31} \text{ kg} \times 0.99 \times 10^8 \text{ m/s}v} \\ &= 6.94 \times 10^{-12} \text{ m} \\ &= 6.94\ p\text{m}. \end{aligned}$$

When we observe electron diffraction, the electrons kinetic energy is easier to measure than their momentum, so we write the de Broglie wavelength in a relativistic notation. From the relativistic formula,

$$\begin{aligned} E^2 &= p^2c^2 + {m_0}^2c^4 \\ \Rightarrow p^2c^2 &= E^2 - m_0^2c^4. \end{aligned} \tag{3}$$

As we know that the total energy of a particle is the sum of its rest mass energy and kinetic energy, therefore,

$$E = m_oc^2 + K.E.,$$

where m_0 = Rest mass of electron = $m_e = 9.1 \times 10^{-31}$ kg, thus equation (**??**) can be written as,

$$\begin{aligned} p^2c^2 &= (m_oc^2 + K.E.)^2 - m_0^2c^4 \\ &= m_0^2c^4 + (K.E.)^2 + 2m_0c^2 \cdot K.E - m_0^2c^4 \\ &= (K.E.)^2 + 2m_0c^2 \cdot K.E. \end{aligned}$$

Kinetic energy of electron is given by,

$$K.E. = \frac{1}{2}m_ev^2,$$

therefore,

$$\begin{aligned} p^2c^2 &= \left(\frac{1}{2}m_ev^2\right)^2 + 2m_0c^2 \times \frac{1}{2}m_ev^2 \\ &= \frac{1}{4}m_e^2v^4 + m_e^2c^2v^2 \\ p^2 &= \frac{m_e^2v^4}{4c^2} + m_e^2v^2 \\ p &= \sqrt{\frac{m_e^2v^4}{4c^2} + m_e^2v^2}. \end{aligned}$$

Thus de Broglie wavelength from equation (??) can be written as,

$$
\begin{aligned}
\lambda &= \frac{h}{p} = \frac{h}{\sqrt{\frac{m_e^2 v^4}{4c^2} + m_e^2 v^2}} \\
&= \frac{2ch}{m_e v \sqrt{v^2 + 4c^2}} \\
&= \frac{2 \times 3 \times 10^8 \text{ m/s} \times 6.63 \times 10^{-34} \text{ Js}}{9.1 \times 10^{-31} \text{ kg} \times 0.99 \times 10^8 \text{ m/s} \sqrt{(0.99 \times 10^8 \text{ m/s})^2 + 4(3 \times 10^8 \text{ m/s})^2}} \\
&= 7.26 \times 10^{-12} \text{ m} \\
&= 7.26 \text{ pm.}
\end{aligned}
$$

(b) i. Now we are given that,

Size of large organic molecule = 10 nm.

Since to observe small objects, by means of diffraction of particles, de Broglie wavelength of the diffracted particle should be comparable to the size of the object, therefore

Wavelength of electron = λ = 10 nm = 10^{-8} m.

Kinetic energy can be calculated by using the equation,

$$K.E. = \frac{1}{2} m_e v^2,$$

where we can calculate speed v of electron by using de Broglie equation,

$$
\begin{aligned}
\lambda &= \frac{h}{p} = \frac{h}{m_e v} \\
\Rightarrow v &= \frac{h}{m_e \lambda}.
\end{aligned}
$$

Thus kinetic energy will become,

$$\begin{aligned} K.E. &= \frac{1}{2}m_e v^2 \\ &= \frac{1}{2}m_e\left(\frac{h}{m_e\lambda}\right)^2 \\ &= \frac{h^2}{2m_e\lambda^2} \\ &= \frac{(6.63\times10^{-34}\ \text{Js})^2}{2\times9.1\times10^{-31}\ \text{kg}\times(10^{-8}\ \text{m})^2} \\ &= 2.4\times10^{-21}\ \text{J} \\ &= \frac{2.4\times10^{-21}}{1.6\times10^{-19}}\ \text{eV} \\ &= 0.015\ \text{eV}. \end{aligned}$$

ii. In this part atomic features of size 10 nm are given. Therefore wavelength of the particle used for diffraction should be equal to 10 nm. As we have already calculated kinetic energy in the above part for 10 nm wavelength that is equal to 0.015 eV. Hence kinetic energy calculations will be same for this part too and the kinetic energy of electron must be 0.015 to observe an atomic feature of 10 nm.

iii. Now we are given that,

$$\text{Size of nucleus} = 10\ \text{fm}.$$

Since to observe small objects, by means of diffraction of particles, de Broglie wavelength of the diffracted particle should be comparable to the size of the object, therefore

$$\text{Wavelength of electron} = \lambda = 10\ \text{fm} = 10\times10^{-15}\ \text{m} = 10^{-14}\ \text{m}.$$

Kinetic energy of the particle will be,

$$\begin{aligned} K.E. &= \frac{h^2}{2m_e\lambda^2} \\ &= \frac{(6.63\times10^{-34}\ \text{Js})^2}{2\times9.1\times10^{-31}\ \text{kg}\times(10^{-14}\ \text{m})^2} \\ &= 2.4\times10^{-9}\ \text{J} \\ &= \frac{2.4\times10^{-9}}{1.6\times10^{-19}}\ \text{eV} \\ &= 1.5\times10^{10}\ \text{eV}. \end{aligned}$$

Now consider the alpha particle. Mass of an alpha particle is 6.64×10^{-27} kg, therefore kinetic energy will be,

$$\begin{aligned} K.E. &= \frac{h^2}{2m_p\lambda^2} \\ &= \frac{(6.63 \times 10^{-34}\ \text{Js})^2}{2 \times 6.64 \times 10^{-27}\ \text{kg} \times (10^{-14}\ \text{m})^2} \\ &= 3.3 \times 10^{-13}\ \text{J} \\ &= \frac{3.3 \times 10^{-13}}{1.6 \times 10^{-19}}\ \text{eV} \\ &= 2.1 \times 10^{6}\ \text{eV}. \end{aligned}$$

(c) We are given that,

Kinetic energy of electron = Kinetic energy of photon = 50 keV = 50×10^3 eV

The wavelengths associated with a photon and an electron of same energy depend on the energy. If kinetic energy of a photon is equal to the energy of an electron and is very very less than the rest mass energy of an electron then the deBroglie wavelength associated with the electron is very very small than the de Broglie wavelength associated with the photon. If energy of a photon is comparable to the rest mass energy of electron they have approximately equal wavelengths. As we know that de Broglie wavelength associated with a particle is given by,

$$\lambda = \frac{h}{p} = \frac{hc}{pc}$$

The relativistic energy expression for a particle is,

$$E^2 = p^2c^2 + m_0^2c^4$$

where photon's rest mass energy is zero, therefore $E^2 = p^2c^2$, $\Rightarrow E = pc$. Thus wavelength associated with photon will be,

$$\lambda_{\text{photon}} = \frac{hc}{E}$$

While the wavelength associated with an electron is given by,

$$\lambda_{\text{electron}} = \frac{h}{p} = \frac{h}{\sqrt{2K_e m_0}},$$

Substitute given values,

$$\begin{aligned}\lambda_{\text{photon}} &= \frac{hc}{E} = \frac{6.63\times10^{-34}\text{ Js}\times3\times10^{8}\text{ m/s}}{5\times10^{4}\text{ eV}}\\ &= \frac{6.63\times10^{-34}\text{ Js}\times3\times10^{8}\text{ m/s}}{5\times10^{4}\times1.6\times10^{-19}\text{ J}}\\ &= 2.48\times10^{-11}\text{ m} = 24.8\text{ pm}.\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_{\text{electron}} &= \frac{h}{\sqrt{2K_e m_0}} = \frac{6.63\times10^{-34}\text{ Js}}{\sqrt{2\times5\times10^{4}\text{ eV}\times9.1\times10^{-31}\text{ kg}}}\\ &= \frac{6.63\times10^{-34}\text{ Js}}{\sqrt{2\times5\times10^{4}\times1.6\times10^{-19}\text{ J}\times9.1\times10^{-31}\text{ kg}}}\\ &= 5.49\times10^{-12}\text{ m} = 5.49\text{ pm}.\end{aligned}$$

(d) We are given that,

$$\begin{aligned}\text{Potential difference} &= V = 10\text{ MV} = 10\times10^{6}\text{ V} = 10^{7}\text{ V}\\ \text{Initial speed of proton} &= v = 0\\ \text{de Broglie wavelength of proton} &= \lambda = ?\end{aligned}$$

Since de Broglie wavelength associated with a charged particle is given by,

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2mqV}}.$$

For proton substitute values,

$$\begin{aligned}\lambda &= \frac{h}{\sqrt{2mqV}}\\ &= \frac{6.63\times10^{-34}\text{ Js}}{\sqrt{2\times1.67\times10^{-27}\text{ kg}\times1.6\times10^{-19}\text{ C}\times10^{7}\text{ V}}}\\ &= 9.06\times10^{-15}\text{ m}\\ &= 9.06\text{ fm}.\end{aligned}$$

2. **Light in a Loop**

Suppose you have an optic fiber wound in a loop of length L. You shoot some light into the loop and it goes round and around in the loop for eternity. Using the idea of continuous wave that we used to explain Bohr quantization of atom (via de Broglie waves), find out what wavelength photons can be sustained inside this loop of optic fiber.

Answer 2

Standing wave patterns are wave patterns produced in a stationary medium when two waves of identical frequencies interfere in such a manner to produce points along the medium that always appear to be standing still. Such standing wave patterns are produced within the medium when it is vibrated at certain frequencies. Each frequency is associated with a different standing wave pattern. These frequencies and their associated wave patterns are referred to as harmonics. A careful study of the standing wave patterns reveal a clear mathematical relationship between the wavelength of the wave that produces the pattern and the length of the medium in which the pattern is displayed. Furthermore, there is a predictability about this mathematical relationship that allows one to generalize and deduce a statement concerning this relationship.

Standing waves can form under a variety of conditions, but they are easily demonstrated in a medium which is finite or bounded. A phone cord begins at the base and ends at the handset. (Or is it the other way around?) Other simple examples of finite media are a guitar string (it runs from fret to bridge), a drum head (it's bounded by the rim), the air in a room (it's bounded by the walls), the water in Lake (it's bounded by the shores), or the surface of the earth (although not bounded, the surface of the earth is finite). In general, standing waves can be produced by any two identical waves traveling in opposite directions that have the right wavelength. In a bounded medium, standing waves occur when a wave with the correct wavelength meets its reflection. The interference of these two waves produces a resultant wave that does not appear to move.

Standing waves are also observed in optical media such as optical wave guides, optical cavities, etc. Lasers use optical cavities in the form of a pair of facing mirrors. The gain medium in the cavity (such as a crystal) emits light coherently, exciting standing waves of light in the cavity. The wavelength of light is very short (in the range of nanometers, 10-9 m) so the standing waves are microscopic in size. One use for standing light waves is to measure small distances, using optical flats.

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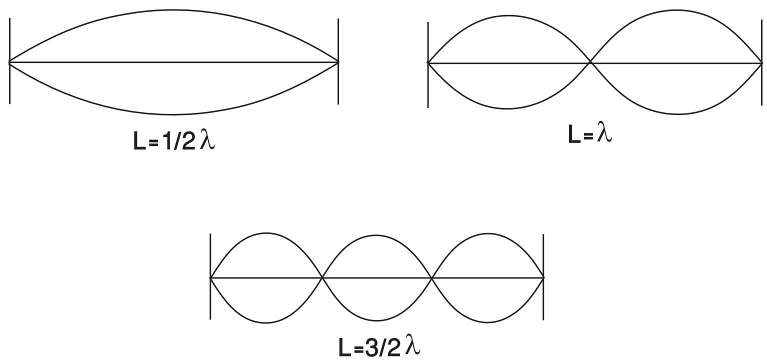
Standing waves are shown in the figure below.



When inspecting the standing wave patterns and the length-wavelength relationships for the first three harmonics, a clear pattern emerges. The number of antinodes in the pattern is equal to the harmonic number of that pattern. The first harmonic has one antinode; the second harmonic has two antinodes; and the third harmonic has three antinodes. Thus, it can be generalized that the nth harmonic has n antinodes where n is an integer representing the harmonic number. The wavelength of standing wave is quantized, can exist which is similar to the idea of continuous wave that we used to explain Bohr quantization of atom (via de Broglie waves). Since in the present case we are dealing with the loop of length L, will be equal to the circumference of the loop, i.e.

$$L = 2\pi r,$$

where r is the radius of the loop. Thus according to Bohr quantization

$$\begin{aligned} mvr &= n\hbar \\ p \times \frac{L}{2\pi} &= n \times \frac{h}{2\pi} \\ \frac{h}{\lambda} \cdot L &= nh \\ \frac{L}{\lambda} &= nh \\ \lambda &= \frac{L}{nh} \end{aligned}$$

3. **Quantum Effects Magnified**

For the purpose of this problem, consider you live in a parallel universe where Planks constant is $\hbar = 10$ Js. You find a very light but strong string of length $\ell = 1$ m, and tie a ball of mass $m = 2$ kg to one end and start rotating this mass in a circle above your head at a constant speed v, by holding the other end of the rope and providing it whatever force is necessary to keep it moving in the circle with speed v.

(a) What would be the De-Broglie wave associated with the moving ball?

(b) What would be the minimum speed that this ball must have when moving in the circle? Recall that the magnitude of the momentum of the ball will be constant when moving with the constant speed, $p = mv$.

(c) If you want to increase its speed, by what minimum amount you must do it?

(d) The ball only has kinetic energy given by $E = \frac{1}{2}mv^2$. What is the energy of the ball in "ground state" and in the "First excited state"?

Ignore the existential questions regarding how your arm (or the rope) will work in such a magnified quantum world.

Answer 3

We are given that,

$$\begin{aligned} \text{Planck's constant} &= \hbar = 10 \text{ Js} \\ \text{Length of string} &= \ell = 1 \text{ m} \\ \text{Mass of ball} &= m = 2 \text{ kg} \\ \text{Speed of whirl of ball string system} &= v. \end{aligned}$$

(a) We want to calculate de Broglie wavelength associated with the moving ball. As wavelength associated with a particle is given by,

$$\begin{aligned}\lambda &= \frac{h}{p} = \frac{h}{mv} = \frac{2\pi\hbar}{mv} \\ &= \frac{2\pi \times 10 \text{ Js}}{2 \text{ kg} \times v \text{ m/s}} \\ &= \frac{10\pi}{v} \text{ m.}\end{aligned}$$

(b) According to Bohr's quantization,

$$\begin{aligned}mvr &= n\hbar \\ v &= \frac{n\hbar}{mr}.\end{aligned}$$

Speed will be minimum when $n = 1$, substitution of given values yield,

$$\begin{aligned}v_0 &= \frac{1 \times 10 \text{ Js}}{2 \text{ kg} \times r} \\ &= \frac{5}{r} \text{ m/s,}\end{aligned}$$

where r is the radius of the circle, which is equal to the length of the string, therefore,

$$\begin{aligned}v_0 &= \frac{5}{1} \text{ m/s} \\ &= 5 \text{ m/s.}\end{aligned}$$

Hence minimum speed of the ball will be 5 m/s.

(c) Since the speed of the ball is quantized and it is the integral multiple of the minimum speed, thus for $n = 2$ speed of the ball will be,

$$\begin{aligned}v_1 &= 2v = 2 \times 5 \text{ m/s} \\ &= 10 \text{ m/s.}\end{aligned}$$

Hence we should increase the speed 5 m/s.

(d) Now we are given that the ball has only kinetic energy $E = \frac{1}{2}mv^2$. Ground state energy can be calculated by using the minimum energy of the ball i.e., speed for

$n = 1$.

$$\begin{aligned}\text{Ground state energy} &= E_0 = \frac{1}{2}mv_0^2 \\ E_0 &= \frac{1}{2} \times 2 \text{ kg} \times (5 \text{ m/s})^2 \\ &= 25 \text{ kg.m}^2\text{s}^{-2} \\ &= 25 \text{ J.}\end{aligned}$$

Similarly energy of the first excited state can be calculated by using the speed of the ball when $n = 2$.

$$\begin{aligned}\text{First excited state energy} &= E_1 = \frac{1}{2}mv_1^2 \\ E_1 &= \frac{1}{2} \times 2 \text{ kg} \times (10 \text{ m/s})^2 \\ &= 100 \text{ kg.m}^2\text{s}^{-2} \\ &= 100 \text{ J.}\end{aligned}$$